



Sliding Mode Vibration Suppression Control Design for a Smart Beam

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ABSTRACT

Active vibration control is the main problem in different structure. Smart material like piezoelectric make a structure smart, adaptive and self-controlling so, they are effective in active vibration control. In this paper piezoelectric elements are used as sensors and actuators in flexible structures for sensing and actuating purposes, and to control the vibration of a cantilever beam by using sliding mode control. The sliding mode controller (SMC) is designed to attenuate the vibration induced by initial tip displacement which is equal to 15 mm. It is designed based on the balance realization reduction method where three states are selected for the reduced model from the 24th states that describe the cantilever beam according to the FEM. These states are most controllable and observable. The stability and control performance for the proposed SMC are proved using candidate Lyapunov function and the equivalent control concept. The control spillover, which is the sources of instability, is completely avoided as ensured within the control performance proof.

Numerical simulations are preformed to test the vibration attenuation ability of the proposed SMC. For 15 mm initial tip displacement, the piezoelectric actuator was found able to reduce the tip displacement to about (0.2) mm within (2.5 s), while it is equal to (3.5) mm with the open loop case. Moreover, the induced chattering in system response, due to the discontinuous control action, is removed by approximating the signum function by a continuous arctan function. As a result a smoother response are obtained with the same control performance as can be shown in the sliding variable, the control input voltage and the tip displacement plots.

Keywords: Active vibration control, Finite Element, sliding mode control, sliding mode observer, spillover.

تصميم مسيطر منزلق الشكل لتخميد الاهتزازات للعتبة الذكية

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الخلاصة

ان السيطرة على الاهتزازات تعتبر من المشاكل الرئيسية في مختلف الهياكل. تستخدم المواد الذكية مثل المواد الكهروضغطية لجعل الهياكل ذكية، ذاتية السيطرة، وهي فعالة في السيطرة على الاهتزازات. في هذه البحث تم استخدام المواد الكهروضغطية كمتحسسات ومؤثرات في الهيكل المرن لغرض السيطرة على الاهتزاز للعتبة باستخدام المسيطر المنزلق الشكل. تم تصميم مسيطر المنزلق الشكل والذي من شأنه تخفيف الاهتزاز الناجم عن ازاحة طرف العتبة بمقدار 15 ملم، تم استخدام طريقة (Balance realization) لاختيار نموذج مخفض حيث يكون هذا النموذج أكثر قابلية على السيطرة والتخمين. تم اختيار ثلاثة متغيرات حالة للنموذج المخفض عن النظام الأصلي والذي كان بعدد 24 متغير. هذه المتغيرات الثلاثة التي تم اختيارها هي الأكثر قابلية على السيطرة والتخمين. الاستقرار وفعالية السيطرة للمسيطر (SMC) المقترح تم اثباته باستخدام (Lyapunov)



function) ومفهوم السيطرة المكافئ. ان تداعيات المسيطر والتي هي من مصادر عدم الاستقرار تم تجنبها بشكل كامل من خلال أداء المسيطر. باستخدام المحاكاة العددية تم اختبار قدرة نظام السيطرة المقترح لتقليل وتخمين الاهتزاز. تم ازالة طرف العتبة 15 مم، حيث وجد بان المؤثرات (actuator) للمادة الكهرضغطية (piz) قادرة على تقليل الازاحة إلى حوالي 0.2 ملم بعد 2.5 ثانية، في حين أنه يساوي 3.5 ملم في الحالة الحرة. أيضا تم التغلب على ظاهرة التذبذب (chattering) باستخدام دالة التقريب لدالة الإشارة عن طريق دالة ظل الزاوية القوسي ونتيجة لذلك تم الحصول على استجابة أكثر سلاسة وبنفس الأداء للمسيطر كما تم مشاهدته بالنسبة لمخططات السطح المنزلق، فولتية المسيطر وكذلك ازالة راس العتبة.

الكلمات الرئيسية: السيطرة الفعالة على الاهتزازات، العناصر المحددة، المسيطر المنزلق الشكل، المخمن المنزلق الشكل، التداعيات

1. INTRODUCTION

The increasing demand of high structural performance requirements has led to the developments of smart materials and structures. A smart structure has the capability to respond to change external environment (such as loads, temperature and shape) as well as to change internal environment (such as damage or failure). This technology has numerous applications, such as active vibration and buckling control, shape control, damage assessment and active noise control. The development of these smart or structures offer great potential or use in advanced aerospace, hydrospace, nuclear, and automotive structural applications, **Bandyopadhyay, 2005**.

The system is called a smart structure because it has the ability to perform self-controlling. One way of making the structure as smart is done by the use of piezoelectric materials. The technology of smart materials and structures especially piezoelectric smart structures has become mature over the last decade. The application of piezoelectric smart structures is the control and suppression of unwanted structural vibrations, **Balamurugan, 2000**.

The main advantages of piezoelectric actuators are fast response, high power density and large force output. Piezoelectric materials can be effectively used for active vibration control with fast response and easy implementation. The electricity for the piezoelectric is produced by pressure (Direct Effect) Conversely, a piezoelectric material deforms when it is subjected to an electric field (Converse Effect). The piezoelectric sensor senses the external disturbances and generates voltage due to direct piezoelectric effect while piezoelectric actuator produces force due to converse piezoelectric effect which can be used as controlling force, **Kumar, et al., 2014**.

To simulate the behavior of mechanical structures under inertia and external loads, very few analytical solutions for specific situations are available. For this reason, the discretization of these structures is the basic step for a static and dynamic further analysis. One possibility for this step is provided by the finite element method. In mathematical terms, finite elements are a numerical method for solving systems, and generally used to eliminate all spatial derivatives by increasing, at the same time, the number of the resulting new equations in the system, **Sachs, 2004**.

The structure is modeled to retain large number of degrees of freedoms. In active vibration control, the use of smaller order model has computational advantages. Therefore, it is necessary to apply a model reduction techniques in order to get a reduced model size for which the control law can be designed. One of these techniques is based on balance realization method, **Inman,**



2006. For closed-loop system, it is not always possible to get a control law that causes eigenvalues to have the required and desired values. This problem raises the concept of controllability. The system is completely controllable if every state variable can be affected in such a way as to cause it to reach a particular value within a finite amount of time by some unbounded control. Then more useful measure is provided for asymptotically stable systems of the form given by equations by defining the controllability gramian. Gramian matrices can be used for checking if a system is controllable and observable, **Zhou, et al., 1999.**

To control vibration of a piezoelectric smart structure, a controller usually designed based on a reduced order model (ROM) of the system form; whereas, finite element models inevitably have a large number of degrees of freedom. When such a ROM based controller is applied to the full order system, actuator forces for reducing the vibration of the lower modes will also influence the residual modes of the structure and produce undesirable vibration due to the unmodeled dynamics. This phenomenon is known as control spillover, **Meirovitch, 1990.** Spillover phenomenon occurs because the unmodeled dynamics, which are not included in reduce order model, will be excited. Different control techniques have been suggested and investigated in the control of smart structure. Some of these studies are linear quadratic regulator (LQR) approach, **Dorf, 2003,** sliding mode control, **Utkin, et al., 2009,** H_2 control, H_∞ control, **Oveisi and Nestorović, 2014.**

The sliding mode control method, first proposed in the early fifties, is one of the control design methods to dominate the uncertainties and disturbances acting on the systems. It is been obtained as significant research attention since early sixties in the former USSR and has been widely applied in a variety of applications, **Bartolini, 2003, Biswas, 2009, Qaiser, 2009.** Sliding mode control (SMC) is a particular type of the so-called Variable Structure Control (VSC) that changes the control direction to drive the system to a specified manifold in the state space and then keep the system within a neighborhood of this manifold. Sliding mode control is designed a controller such that the motion of the system tends to slide mode surface. Therefore designing a SMC consists of two stages; finding a sliding surface (defined as a desired linear combination of system states such as displacement, velocity, and acceleration) to stabilize the controlled system, and find a control force to drive the response trajectory into the sliding surface with an exponential speed in time, **Itik and Salamci, 2005.**

The main feature of sliding mode control is its insensitivity to some class of uncertainties, which makes it attractive in the control applications for uncertain systems. The sliding mode control method has some advantages such as robustness to parameter uncertainty, insensitivity to bounded disturbances, fast dynamic response, and easy implementation of the controller, **Magnani, 2007, Ferrara and Vecchio, 2009, and Capisani, 2009.** The method enables the decoupling of overall system motion into independent partial components of low dimension and as a result reduces the complexity of feedback design Sliding mode theory has been recognized as a robust control approach in treating disturbances and modeling uncertainties through the concepts of sliding surface design and equivalent control, **Utkin, et al., 2009.**

The aim of the present paper is to design sliding mode control to attenuate the vibration of a smart cantilever beam using piezoelectric element. The model utilized for control design purpose is the reduced order model that is obtained according to the balance realization method. Based on the equivalent control the performance of the proposed SMC is ensured via satisfying the



control performance condition. Consequently the control spillover is eliminated by satisfying this condition.

2. SMART CANTILEVER SYSTEM MODEL

The model of cantilever flexible beam studied here is given in **Fig.1**. The cantilever beam bonded with the same place pair of piezoelectric sensor / actuator near the fixed end. By using the Euler-Bernoulli beam equation, the infinite dimensional mathematical expression of the beam can be written as follows, **Bandyopadhyay, 2007**.

$$c^2 \frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad (1)$$

where $c^2 = EI/\rho A$, $w(x,t)$ is the deflection along the x -axis, E is the Young's modulus, I is the moment of inertia, A is the cross sectional area, and ρ is the density of the beam. The partial differential equation (PDE) given by Eq. (1) can be solved by using the supposed mode approach, which yields finite dimensional ordinary differential equation set.

The dynamic equation of the smart structure is obtained by using both regular beam element and piezoelectric beam elements. The mass and stiffness matrices of the smart structure include sensor/actuator mass and stiffness, **Chhabra, et al., 2012**. The entire structure is modelled in state space form using the Finite Element Method (FEM) by dividing the structure into six equal finite elements. The sensor and actuator were integrated on the top and bottom surfaces at the second element from the fixed end of the beam. A beam element is considered with two nodes at its end. Each node having two degree of freedom (DOF) (translation and rotation) is considered. The mass and stiffness matrix is derived using shape functions for the beam element. When a system vibrates, it undergoes back and forth motion, it has transverse displacements, so all positions vary with time, and therefore, the system has velocities and accelerations. The equation of motion, involves a fourth order derivative w.r.t.(x) and a second order derivative w.r.t. time (acceleration) The solution of the Eq. (1) is assumed as a cubic polynomial function of (x) given by:

$$w(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \quad (2)$$

where $w(x)$ is displacement function which satisfies the fourth order partial differential equation (1). The constants a_1 to a_4 are obtained by using the boundary conditions given below at both the nodal points (fixed end and free end). Consider the derivative of $w(x)$ as:

$$\frac{\partial w}{\partial x} = a_2 + 2a_3x + 3a_4x^2 \quad (3)$$

then at $x = 0$, $w(x) = a_1 = w_1$ and $\frac{\partial w}{\partial x} = a_2 = \theta_1$. Also at $x = l$,
 $w(x) = w_2 = a_1 + a_2l + a_3l^2 + a_4l^3$, and $\frac{\partial w}{\partial x} = \theta_2 = a_2 + 2a_3l + 3a_4l^2$

where $w_1, \theta_1, w_2, \theta_2$ are Degree of Freedom at node 1 and 2, respectively and l is the length of the regular beam. The relation between $w_1, \theta_1, w_2, \theta_2$ and the constants a_1 to a_4 is represented in a matrix form as,



$$\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad (4)$$

Solving for a_1 to a_4 yields;

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \frac{1}{l^3} \begin{bmatrix} l^3 & 0 & 0 & 0 \\ 0 & l^3 & 0 & 0 \\ -3l & -2l^2 & 3l & -l^2 \\ 2 & l & -2 & l \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} \quad (5)$$

Substituting the constants obtained from (5) into (2) and by rearranging the terms, the final form for $w(x)$ is obtained as:

$$w(x) = \frac{1}{l^3} [(l^3 - 3lx^2 + 2x^3) (l^3x - 2l^2x^2 + x^3l) (3lx^2 - 2x^3) (-l^2x^2 + lx^3)] \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} \quad (6)$$

$$\text{or } w(x) = N^T q \quad (7)$$

where N is the shape function and q is the displacements at the nodes, which are given by

$$N = \frac{1}{l^3} [(l^3 - 3lx^2 + 2x^3) (l^3x - 2l^2x^2 + x^3l) (3lx^2 - 2x^3) (-l^2x^2 + lx^3)]^T \quad (8)$$

$$q = [w_1 \ \theta_1 \ w_2 \ \theta_2]^T \quad (9)$$

The strain energy U and the kinetic energy T for the beam element with uniform cross section in bending is obtained as:

$$U = \frac{E_b I_b}{2} \int_{l_b} \left[\frac{\partial^2 w}{\partial x^2} \right]^2 dx = \frac{E_b I_b}{2} \int_{l_b} [w''(x, t)]^T [w''(x, t)] dx \quad (10)$$

$$T = \frac{\rho_b A_b}{2} \int_{l_b} \left[\frac{\partial w}{\partial t} \right]^2 dx = \frac{\rho_b A_b}{2} \int_{l_b} [\dot{w}(x, t)]^T [\dot{w}(x, t)] dx \quad (11)$$

where ρ_b is the mass density of the beam material, A_b is the cross sectional area of the beam, I_b is the moment of inertia of the beam, and E_b is the modulus of elasticity of the beam material. The equation of motion of the regular beam element is obtained by using the Lagrangian equation:

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_i} \right] + \left[\frac{\partial U}{\partial q_i} \right] = [F_i] \quad (12)$$



For free vibration, $F_i = 0$. The strain energy U and the kinetic energy T in terms of the shape function N and q are

$$\begin{aligned} T &= \frac{\rho_b A_b}{2} \int_{l_b} [N^T \dot{q}]^T [N^T \dot{q}] dx = \frac{\rho_b A_b}{2} \int_{l_b} \dot{q}^T N N^T \dot{q} dx \\ &= \dot{q}^T \left(\frac{\rho_b A_b}{2} \int_{l_b} N N^T dx \right) \dot{q} = \frac{1}{2} \dot{q}^T M_b \dot{q} \end{aligned} \quad (13)$$

Accordingly

$$\frac{\partial T}{\partial \dot{q}_i} = M_b \dot{q} \quad (14)$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_i} \right] = M_b \ddot{q} \quad (15)$$

where M_b is the mass matrix of regular beam

$$M_b = \rho_b A_b \int_{l_b} N N^T dx = \frac{\rho_b A_b l_b}{420} \begin{bmatrix} 156 & 22l_b & 54 & -13l_b \\ 22l_b & 4l_b^2 & 13l_b & -3l_b^2 \\ 54 & 13l_b & 156 & -22l_b \\ -13l_b & -3l_b^2 & -22l_b & 4l_b^2 \end{bmatrix} \quad (16)$$

Also for the strain energy,

$$\begin{aligned} U &= \frac{E_b I_b}{2} \int_{l_b} [N_2^T q]^T [N_2^T q] dx = \frac{E_b I_b}{2} \int_{l_b} q^T N_2 \cdot N_2^T q dx \\ &= q^T \left(\frac{E_b I_b}{2} \int_{l_b} N_2 N_2^T dx \right) q = \frac{1}{2} q^T K_b q \end{aligned} \quad (17)$$

one can obtain

$$\left[\frac{\partial U}{\partial q_i} \right] = K_b q \quad (18)$$

$$K_b = E_b I_b \int_{l_b} N_2 N_2^T dx \quad (19)$$

where K_b is the stiffness matrix of regular beam

$$K_b = E_b I_b \int_{l_b} N_2 N_2^T dx = \frac{E_b I_b}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad (20)$$

Eventually the equation of motion according to the Lagrangian equation is:

$$M_b \ddot{q} + K_b q = f_b \quad (21)$$

or



$$\frac{\rho_b A_b}{420} \begin{bmatrix} 156 & 22l_b & 54 & -13l_b \\ 22l_b & 4l_b^2 & 13l_b & -3l_b^2 \\ 54 & 13l_b & 156 & -22l_b \\ -13l_b & -3l_b^2 & -22l_b & 4l_b^2 \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \frac{E_b I_b}{l^3} \begin{bmatrix} 12 & 6l_b & -12 & 6l_b \\ 6l_b & 4l_b^2 & -6l_b & 2l_b^2 \\ -12 & -6l_b & 12 & -6l_b \\ 6l_b & 2l_b^2 & -6l_b & 4l_b^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} \quad (22)$$

where F_1, F_2, M_1, M_2 are the forces and the bending moments acting on nodes 1 and 2 respectively **Fig.1**. When PZT patches are assumed as Euler-Bernoulli beam elements the elemental mass and stiffness matrices of PZT beam element can be computed in similar fashion as, **Bandyopadhyay, 2007**.

$$M_b = \frac{\rho_p A_p l_p}{420} \begin{bmatrix} 156 & 22l_p & 54 & -13l_p \\ 22l_p & 4l_p^2 & 13l_p & -3l_p^2 \\ 54 & 13l_p & 156 & -22l_p \\ -13l_p & -3l_p^2 & -22l_p & 4l_p^2 \end{bmatrix} \quad (23)$$

$$K_b = \frac{E_p I_p}{l^3} \begin{bmatrix} 12 & 6l_p & -12 & 6l_p \\ 6l_p & 4l_p^2 & -6l_p & 2l_p^2 \\ -12 & -6l_p & 12 & -6l_p \\ 6l_p & 2l_p^2 & -6l_p & 4l_p^2 \end{bmatrix} \quad (24)$$

The smart beam element is obtained by sandwiching the regular beam element in between the two PZT patches **Fig. 1**.

In which $EI = E_b I_b + 2E_p I_p$ is the flexural rigidity and $\rho A = b(\rho_b t_b + 2\rho_p t_p)$ is the mass per unit length of smart beam element, t_p is the thickness of PZT patches thickness of Actuator and Sensor, and $I_p = \frac{b t_a^3}{12} + b t_a \left(\frac{t_a + t_b}{2}\right)^2$. So the elemental mass and stiffness matrices of smart beam element are:

$$M_e = \frac{\rho A l_b}{420} \begin{bmatrix} 156 & 22l_b & 54 & -13l_b \\ 22l_b & 4l_b^2 & 13l_b & -3l_b^2 \\ 54 & 13l_b & 156 & -22l_b \\ -13l_b & -3l_b^2 & -22l_b & 4l_b^2 \end{bmatrix} \quad (25)$$

$$K_e = \frac{EI}{l_p^3} \begin{bmatrix} 12 & 6l_b & -12 & 6l_b \\ 6l_b & 4l_b^2 & -6l_p & 2l_b^2 \\ -12 & -6l_b & 12 & -6l_b \\ 6l_b & 2l_b^2 & -6l_b & 4l_b^2 \end{bmatrix} \quad (26)$$



2.1 Sensor and Actuator Equations

The sensor equation is derived from the direct piezoelectric equation, which is used to calculate the total charge created by the strain in the structure. Piezoelectric materials can be used as strain rate sensors. When used so, the output charge can be transformed into the sensor current $i(t)$, **Bandyopadhyay, 2007**.

$$i(t) = z e_{31} b \int_{x_i}^{x_i+l_b} N_2^T \dot{q} dx \quad (27)$$

where, $z = \frac{t_b}{2} + t_a$ and N_2 is the second spatial derivative of the shape function, e_{31} is the piezoelectric stress constant.

The output current of the piezoelectric sensor measures the moment rate of the flexible beam. This current is converted into the open circuit sensor voltage $V_s(t)$ using a signal-conditioning device with the gain G_c , **Bandyopadhyay, 2007**.

$$V_s(t) = [0 \quad -G_c z e_{31} b \quad 0 \quad G_c z e_{31} b] \begin{bmatrix} \dot{w}_1 \\ \dot{\theta}_1 \\ \dot{w}_2 \\ \dot{\theta}_2 \end{bmatrix} = S_c [0 \quad -1 \quad 0 \quad 1] \begin{bmatrix} \dot{w}_1 \\ \dot{\theta}_1 \\ \dot{w}_2 \\ \dot{\theta}_2 \end{bmatrix} = p^T \dot{q} \quad (28)$$

where $S_c = G_c z e_{31} b$ and p is a constant vector depends on the type of sensor, its characteristics and its location on the beam. The actuator equation is derived from the converse piezoelectric equation. The strain developed ϵ_x by the electric field E_f on the actuator layer is given by, **Jalili, 2010**.

$$\epsilon = dE_f \quad (29)$$

where, $E_f = \frac{V_a(t)}{t_a}$ is the electric field, and $V_a(t)$ is the input voltage applied to the piezoelectric actuator in the thickness direction t_a . Then the stress σ_a that developed by the actuator is given by, **Bandyopadhyay, 2007**.

$$\sigma_a = E_p d_{31} \left(\frac{V_a(t)}{t_a} \right) \quad (30)$$

where E_p is the Young's modulus of the piezoelectric and d_{31} is piezoelectric strain constant. The bending moment in a small cross section of the piezoelectric element is given by:

$$dM_a = E_p I_p \frac{d^2 w}{dx^2} \quad (31)$$

The resultant moment M_a acting on the beam element due to the applied voltage V_a is determined by integrating the stress in Eq. (30) throughout the structure thickness as:

$$M_a = E_p d_{31} z V_a(t) \quad (32)$$



The control force f_{ctrl} produced by the actuator that is applied on the beam element is obtained as, **Bandyopadhyay, 2007**.

$$f_{ctrl} = E_p d_{31} b z [-1 \ 0 \ 1 \ 0]^T V_a(t) \quad (33)$$

Alternatively, f_{ctrl} can be expressed as:

$$f_{ctrl} = h V_a(t) \quad (34)$$

where,

$$h = E_p d_{31} b z [-1 \ 0 \ 1 \ 0]^T \quad (35)$$

3. DYNAMIC EQUATION OF SMART STRUCTURE

The dynamic equation of the smart structure is obtained by using both the regular and piezoelectric beam elements (local matrices) given by Eq. (25) and Eq. (26). The mass and stiffness of the bonding or the adhesive between the master structure and the sensor / actuator pair is neglected. The mass and stiffness of the entire beam, which is divided into six finite elements with the piezo-patches placed at only one discrete location is assembled using the FEM technique and the assembled matrices (global matrices) M and K are obtained. The equation of motion of the smart structure is given by, **Bandyopadhyay, 2007**.

$$M \ddot{q} + Kq = f_{ext} + f_{ctrl} = f \quad (36)$$

where M , K , f_{ext} , f_{ctrl} and f are the global mass matrix, global stiffness matrix of the smart beam, the external force applied to the beam, the controlling force from the actuator and the total force coefficient vector respectively.

The generalized structural modal damping matrix D is introduced into Eq. (36) by using, **Balamurugan and Narayanan, 2000, Clough, 2007**.

$$D = \alpha M + \beta K \quad (37)$$

where α and β are the frictional damping constant and the structural damping constant respectively. When applying the cantilever beam boundary condition, the system equation of motion for the 6-element cantilever beam is:

$$M \ddot{q} + D\dot{q} + Kq = f \quad (38)$$

For free vibration condition f_{ext} equal to zero, so the remaining applied force on the system is the controlling force f_{ctrl} exerted by the controller.

3.1 State Space Model of the Smart Structure

Many design tools and model reduction in modern control theory need a state space form for the mathematical model of a plant. Consequently, the smart flexible cantilever beam mathematical model can be written in state space form as follows;



Let $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$, $\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$, and $\ddot{q} = \begin{bmatrix} \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix}$ then the 6-element smart cantilever beam state space model is;

$$M \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + D \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f_{ctrl} \quad (39)$$

which yields

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = -M^{-1}D \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - M^{-1}K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + M^{-1}h V_a(t) \quad (40)$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}h \end{bmatrix} V_a(t) \quad (41)$$

And in a matrix form

$$\dot{x} = Ax(t) + Bu(t) \quad (42)$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ M^{-1}h \end{bmatrix}$ and $u(t) = V_a(t)$.

with appropriate zero and identity matrices dimensions. The sensor voltage is taken as the output of the system and the output equation is obtained as:

$$y(t) = V_s(t) = p^T \dot{q} = p^T \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \quad (43)$$

Thus, the sensor output equation in state space form is given by:

$$y(t) = \begin{bmatrix} 0 & p^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (44)$$

or,

$$y(t) = Cx(t) \quad (45)$$

where $C = \begin{bmatrix} 0 & p^T \end{bmatrix}$. The single input single output state space model (state equation and the output equation) of the smart structure developed for the system is given by Eqs. (42) and (45):

$$\left. \begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y &= Cx(t) \end{aligned} \right\} \quad (46)$$



With

$$\left. \begin{aligned} A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}_{24 \times 24} \\ B &= \begin{bmatrix} 0 \\ M^{-1}h \end{bmatrix}_{24 \times 1} \\ C &= [0 \quad p^T]_{1 \times 24} \end{aligned} \right\} \quad (47)$$

In the following section, the state space model is reduced via balance realization to a form and dimension more appropriate for controller and observer design.

3.2 Model Reduction

In the finite element modeling, the structure is modeled to retain large number of degrees of freedoms. In active vibration control, the use of smaller order model has computational advantages. Therefore, it is necessary to apply a model reduction technique to the state space representation. The reduced order system model extraction techniques solve the problem of the complexity by keeping the essential properties of the full model only, **Inman, 2006**. For the present work the 24th order system model obtained from the finite element model is reduced to the three order using a model reduction technique based on balance realization. The approach taken for reduction the order of a given model based on deleting the coordinates, or modes, that are the least controllable and observable. To implement this idea, a measure of the degree of controllability and observability is needed. However, an alternative, more useful measure is provided for asymptotically stable systems of the form given by equations by defining the controllability grammian, denoted by W_C , as

$$W_C^2 = \int_0^{\infty} e^{At} B B^T e^{A^T t} dt \quad (48)$$

And the observability grammian, denoted by W_O , as , **Inman, 2006**.

$$W_O^2 = \int_0^{\infty} e^{A^T t} C^T C e^{At} dt \quad (49)$$

The matrices A , B , and C defined as in Eq. (47). The properties of these matrices provide useful information about the controllability and observability of the closed-loop system. If the system is controllable (or observable), the matrix W_C (or W_O) is nonsingular, **Williams and Lawrence, 2007**. These grammians characterize the degree of controllability and observability by quantifying just how far away from being singular the matrices W_C and W_O are, **Janardhanan, 2013**.

Applying the idea of singular values as a measure of rank deficiency to the controllability and observability grammians yields a systematic model reduction method. The matrices W_C and W_O are symmetric and hence are similar to a diagonal matrix. There is equivalent system for which these two grammians are both equal and diagonal. Such a system is called balanced system, also W_C and W_O must satisfy the two Liapunov-type equations:



$$\left. \begin{aligned} A W_C^2 + W_C^2 A^T &= -BB^T \\ A^T W_O^2 + W_O^2 A &= -C^T C \end{aligned} \right\} \tag{50}$$

Now to transform the system to a balance realization form, this requires the determination of a transformation matrix P that will transform the system in Eq. (46) to:

$$\left. \begin{aligned} \dot{x}' &= A'x' + B'u \\ y &= C'x' + Du \end{aligned} \right\} \tag{51}$$

where $A' = P^{-1}AP$, $B' = P^{-1}B$ and $C' = CP$. The controllability and observability grammians matrices are diagonal and equal to

$$\widehat{W}_C = \widehat{W}_O = \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$$

where \widehat{W}_C and \widehat{W}_O are the controllability and observability grammians for system after applying the transformation P and the numbers σ_i are the singular values of the grammians and are ordered such that, $\sigma_i > \sigma_{i+1}, i = 1, 2, \dots, n$

Therefore the pair (A', B') could be uncontrollable pair since some of σ_j could be equal to zero. Indeed there exists a subsystem (i.e., a reduced order model) which is still controllable and observable.

Now the choice

$$P = G^{-1}U\Sigma^{\frac{1}{2}} \tag{52}$$

will transform the grammians W_C^2 and W_O^2 to become equal and transform the system in Eq. (46) to a balanced realization form. Namely,

$$\widehat{W}_C = \widehat{W}_O = \Sigma \tag{53}$$

where Σ can be written in terms of two set of the singular values $\sigma_{(1)}$ and $\sigma_{(2)}$ as

$$\Sigma = \begin{bmatrix} \sigma_{(1)} & 0 \\ 0 & \sigma_{(2)} \end{bmatrix} \tag{54}$$

In this representation $\sigma_{(1)}$ describes the “strong” sub-systems to be retained and $\sigma_{(2)}$ the “weak” sub-systems to be deleted. Conformally partitioning the matrices as

$$\left. \begin{aligned} A' &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ B' &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ C' &= [C_1 \quad C_2] \end{aligned} \right\} \tag{55}$$

and truncating the model, retaining $A_{1r} = A_{11}$, $B_r = B_1$ and $C_r = C_1$ as the reduced system, and deleting the “weak” internal subsystems, **Inman, 2006.**



4. SLIDING MODE CONTROL DESIGN

This section is devoted to design a sliding mode controller to the smart cantilever beam using its reduced order model. The sliding mode control approach is recognized as one of the efficient tools to design robust controllers for complex high-order nonlinear dynamical systems which are operating under parameter's uncertainty or in presence of disturbance inputs, **Al-khazraji and Hamzaoui, 2006**. Sliding mode theory has been recognized as a robust control approach in treating disturbances and modeling uncertainties through the concepts of sliding surface design and equivalent control. The equivalent control method means replacement of discontinuous control on the intersection of switching surfaces by a continuous one such that the state velocity vector lies in the tangential manifold, **Utkin, et al., 2009**.

The major advantage of the sliding mode control design approach is the low sensitivity to the system model parametric variations and disturbances which eliminates the necessity of exact modeling, **Bandyopadhyay, 2005**. The sliding mode design method consists of two steps. The first step, a sliding surface is designed so that the state trajectory of the plant forced to the required surface, and the second step is to design a control law such that the system remains on the sliding surface. Therefore, the design of SMC includes the determination of sliding surface and controller design, **Balamurugan and S. Narayanan, 2000**.

To design a sliding mode control to the reduced order model of the smart beam the Reduced Model (RM) and the Residual Model (RSM), **Dorf, 2003**, are presented here as follows; according to the balance realization the linear state model for the cantilever beam, as given in Eq. (51) are rewritten as follow;

$$\left. \begin{aligned} \dot{x}_r &= A_{1r}x_r + A_{1R}x_R + B_1u \\ \dot{x}_R &= A_{2r}x_r + A_{2R}x_R + B_2u \\ y &= C_r x_r + C_R x_R \end{aligned} \right\} \quad (56)$$

where $x_r \in \mathcal{R}^r$, is the *reduced model states*, $x_R \in \mathcal{R}^{n-r}$ is the *residual model states*, and

$$\begin{aligned} A &= \begin{bmatrix} A_{1r}^{r \times r} & A_{1R}^{r \times (n-r)} \\ A_{2r}^{(n-r) \times r} & A_{2R}^{(n-r) \times (n-r)} \end{bmatrix}, \\ B &= \begin{bmatrix} B_1^{r \times 1} \\ B_2^{(n-r) \times 1} \end{bmatrix}, \\ C &= [C_r^{1 \times r} \quad C_R^{1 \times (n-r)}] \end{aligned}$$

where the pair (A_{1r}, B_1) is a controllable pair with highest controllability and observability grammian. The RM of Eq. (51) from Eq. (56) is

$$\dot{x}_r = A_{1r}x_r + A_{1R}x_R + B_1u \quad (57)$$

In order to design a SMC for the reduced model, and as a first step, it is required to transform Eq. (57) to the so-called Regular Form (RF). The sliding mode control had two-stage design procedure which is the selection a switching manifold and then finding control enforcing sliding mode in this manifold, these two stage becomes simpler for systems in RF. The regular form consists of two blocks; the first block does not depend on control, whereas the dimension of



the second block coincides with that of the control, **Utkin, et al., 2009**. The RM is decomposed first to the form;

$$\begin{cases} \dot{x}_{r1} = A_{1r11}x_{r1} + A_{1r12}x_{r2} + A_{1R1}x_R + B_{11}u \\ \dot{x}_{r2} = A_{1r21}x_{r1} + A_{1r22}x_{r2} + A_{1R2}x_R + B_{12}u \end{cases} \quad (58)$$

where:

$$x_r = \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix}, x_{r1} \in \mathcal{R}^{r-\alpha}, x_{r2} \in \mathcal{R}^\alpha$$

$$A_{1r} = \begin{bmatrix} A_{1r11}^{(r-\alpha) \times (r-\alpha)} & A_{1r12}^{(r-\alpha) \times \alpha} \\ A_{1r21}^{\alpha \times (r-\alpha)} & A_{1r22}^{\alpha \times \alpha} \end{bmatrix}, A_{1R} = \begin{bmatrix} A_{1R1} \\ A_{1R2} \end{bmatrix} \text{ and } B_1 = \begin{bmatrix} B_{11}^{(r-\alpha) \times \alpha} \\ B_{12}^{\alpha \times \alpha} \end{bmatrix}$$

The required transformation matrix to the RF is presented in the following proposition.

Proposition (1) The RM as given in Eq. (58) is transformed to the RF via the following transformation;

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T_r x_r = \begin{bmatrix} I_{(r-\alpha)} & -B_{11}B_{12}^{-1} \\ O_{\alpha \times (r-\alpha)} & I_\alpha \end{bmatrix} \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} \quad (59)$$

where I and O are the identity and zero matrices respectively with the matrix size given as the subscript.

Proof: The validity of the transformation T_r can be proved as follows; first it is needed to show that T_r is a nonsingular matrix and then to show that the RM is transformed to the regular form via T_r . The transformation matrix T_r is nonsingular since, **Bernstein,2009**.

$$\begin{aligned} \det(T_r) &= \det \begin{bmatrix} I_{(r-\alpha)} & -B_{11}B_{12}^{-1} \\ O_{\alpha \times (r-\alpha)} & I_\alpha \end{bmatrix} \\ &= \det I_{(r-\alpha)} \det I_\alpha = 1 \end{aligned}$$

Secondly, the RM in terms of the new state z is;

$$\dot{z} = T_r A_{1r} T_r^{-1} z + T_r A_{1R} x_R + T_r B_1 u \quad (60)$$

All what it is necessary to prove it is that the control term $T_r B_1 u$ doesn't appear in \dot{z}_1 . Namely;

$$T_r B_1 = \begin{bmatrix} I_{(r-\alpha)} & -B_{11}B_{12}^{-1} \\ O_{\alpha \times (r-\alpha)} & I_\alpha \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = \begin{bmatrix} O_{(r-\alpha) \times \alpha} \\ \hat{B}_{12} \end{bmatrix}$$

Accordingly the RM dynamics becomes (The RF model);

$$\begin{cases} \dot{z}_1 = \hat{A}_{r11}z_1 + \hat{A}_{r12}z_2 + \hat{A}_{R1}x_R \\ \dot{z}_2 = \hat{A}_{r21}z_1 + \hat{A}_{r22}z_2 + \hat{A}_{R2}x_R + \hat{B}_{12}u \end{cases} \quad (61)$$

Where



$$\hat{A}_{1r} = T_r A_{1r} T_r^{-1} = \begin{bmatrix} \hat{A}_{r11} & \hat{A}_{r12} \\ \hat{A}_{r21} & \hat{A}_{r22} \end{bmatrix}, \text{ and}$$

$$\hat{A}_{1R} = T_r A_{1R} = \begin{bmatrix} \hat{A}_{R1} \\ \hat{A}_{R2} \end{bmatrix}$$

Remark 1: By considering the transformation matrix T_r , which it is devoted to the RM state only, one can easily find the overall **non-singular** transformation matrix to system model state x (Eq. (51)) in the form

$$\hat{x} = \begin{bmatrix} z_1 \\ z_2 \\ x_R \end{bmatrix} = T_{rT} x = \begin{bmatrix} T_r^{r \times r} & O_{r \times (n-r)} \\ O_{(n-r) \times r} & I_{(n-r) \times (n-r)} \end{bmatrix} x \tag{62}$$

And accordingly Eq. (51) is transformed to

$$\dot{\hat{x}} = T_{rT} A T_{rT}^{-1} \hat{x} + T_{rT} B u$$

Or

$$\left. \begin{aligned} \dot{z}_1 &= \hat{A}_{r11} z_1 + \hat{A}_{r12} z_2 + \hat{A}_{R1} x_R \\ \dot{z}_2 &= \hat{A}_{r21} z_1 + \hat{A}_{r22} z_2 + \hat{A}_{R2} x_R + B_{12} u \\ \dot{x}_R &= \hat{A}_{R3} z_1 + \hat{A}_{R3} z_2 + \hat{A}_{R3} x_R + B_2 u \end{aligned} \right\} \tag{63}$$

Equation (63) can be named as the Total Regular Form (TRF) model where the RF model (Eq. (61)) is the upper part of it.

Note (1) that $\det(T_{rT}) = \det(T_r) = 1$ and the total transformation from x to \hat{x} is

$$\hat{x} = T_{rT} T_o x = T x \tag{64}$$

where

$$T = T_{rT} T_o \tag{65}$$

$$T_o = P^{-1}$$

Remark 2: In the RF model in Eq. (61) the terms $\hat{A}_{1R2} x_R$ and $\hat{A}_{1R1} x_R$ are the **matched** and **unmatched** disturbances, **Castaños and Fridman,2006**.

Proposition (2): For the RF model in Eq. (61) with $\alpha = 1$, the sliding mode controller that will regulate the system state x to the origin is given by

$$u = u_0 + u_s \tag{66}$$

where

$$s = z_2 + G z_1, G \in \mathcal{R}^{1 \times (r-1)} \tag{67}$$

$$u_0 = -B_{12}^{-1} (\hat{A}_{r21} z_1 + \hat{A}_{r22} z_2 + G \hat{A}_{r11} z_1 + G \hat{A}_{r12} z_2) \tag{68}$$

$$u_s = k * \text{sgn}(s) \tag{69}$$

$$k = |B_{12}|^{-1} |\hat{A}_{R2} + G \hat{A}_{R1}| * \sup_{t \geq 0} |x_R| + k_o, k_o > 0 \tag{70}$$

With the selection of the matrix G such that;

- i. the matrix $(\hat{A}_{r11} - \hat{A}_{r12} G)$ is Hurwitz
- ii. the matrix $(A + BH)$ must has $(n - 1)$ negative roots plus one equal to zero value where

$$H = -B_{12}^{-1} [(\hat{A}_{r21} + G \hat{A}_{r11}) \quad (\hat{A}_{r22} + G \hat{A}_{r12}) \quad (\hat{A}_{R2} + G \hat{A}_{R1})] T \tag{71}$$



iii. the following *control performance condition* is satisfied;

$$R^C = \min_{i=1 \rightarrow n} |R_i^C| > \min_{i=1 \rightarrow n} |R_i^O| = R^O \quad (72)$$

Where R_i^C represents the real term of the i^{th} eigenvalue of $(A + BH)$ except zero and R_i^O represents the real term of the i^{th} eigenvalue of A . The zero eigenvalue of $(A + BH)$ is due to constrain the system state to the sliding manifold $s = 0$.

Proof: the objective of the SMC is to direct the sliding variable s to the origin using a discontinuous control action. Accordingly, a SMC is designed such that it makes the derivative of the candidate Lyapunov function $V(s) = \frac{1}{2}s^2$ negative definite. The time derivative of V is

$$\dot{V}(s) = s * \dot{s}$$

and the time derivative of s is

$$\dot{s} = \dot{z}_2 + G\dot{z}_1 = \hat{A}_{r21}z_1 + \hat{A}_{r22}z_2 + \hat{A}_{R2}x_R + B_{12}u + G\hat{A}_{r11}z_1 + G\hat{A}_{r12}z_2 + G\hat{A}_{R1}x_R$$

Select the control law as

$$u = u_0 + u_s$$

Where u_0 and u_s are the nominal (continuous) and the discontinuous control terms, the \dot{s} then becomes;

$$\begin{aligned} \dot{s} &= \hat{A}_{r21}z_1 + \hat{A}_{r22}z_2 + G\hat{A}_{r11}z_1 + G\hat{A}_{r12}z_2 + B_{12}u_0 + B_{12}u_s + \hat{A}_{R2}x_R + G\hat{A}_{R1}x_R \\ &= B_{12}u_s + (\hat{A}_{R2} + G\hat{A}_{R1})x_R \end{aligned}$$

Where the control term u_0 is selected to eliminate the known terms in \dot{s} as;

$$u_0 = -B_{12}^{-1}(\hat{A}_{r21}z_1 + \hat{A}_{r22}z_2 + G\hat{A}_{r11}z_1 + G\hat{A}_{r12}z_2)$$

The term $(\hat{A}_{R2} + G\hat{A}_{R1})x_R$ is the unknown term due to the unmeasurable (or estimated) states x_R and for which u_s is devoted as follows;

$$u_s = k * \text{sgn}(s)$$

To this end \dot{s} becomes;

$$\begin{aligned} \dot{s} &= B_{12} * k * \text{sgn}(s) + (\hat{A}_{R2} + G\hat{A}_{R1})x_R \\ &= -|B_{12}| * k * \text{sgn}(s) + (\hat{A}_{R2} + G\hat{A}_{R1})x_R \end{aligned}$$

where $B_{12} < 0$. The discontinuous gain k is determined such that $\dot{V}(s) < 0$. So let k be given by

$$k = |B_{12}|^{-1}|\hat{A}_{R2} + G\hat{A}_{R1}| * \sup_{t \geq 0} |x_R| + k_o, \quad k_o > 0$$

Then $\dot{V}(s)$ becomes

$$\begin{aligned} \dot{V}(s) &= s\dot{s} = s\{-|B_{12}| * k * \text{sgn}(s) + (\hat{A}_{R2} + G\hat{A}_{R1})x_R\} \\ &= -|s||B_{12}| * k + s(\hat{A}_{R2} + G\hat{A}_{R1})x_R \\ &< -|s||B_{12}| * k + |s| |(\hat{A}_{R2} + G\hat{A}_{R1})x_R| = -|s|(|B_{12}| * k - |\hat{A}_{R2} + G\hat{A}_{R1}||x_R|) \\ &= -|s|\{|B_{12}| * (|B_{12}|^{-1}|\hat{A}_{R2} + G\hat{A}_{R1}| * \sup_{t \geq 0} |x_R| + k_o) - |\hat{A}_{R2} + G\hat{A}_{R1}||x_R|\} \\ &= -|s|\{k_o + |\hat{A}_{R2} + G\hat{A}_{R1}|(\sup_{t \geq 0} |x_R| - |x_R|)\} \end{aligned}$$

Since $(\sup_{t \geq 0} |x_R| - |x_R|) > 0 \forall t$, then $\dot{V}(s) < 0$. This will lead to $s = \dot{s} = 0$ after a finite interval of time. To this end, the attractiveness of the sliding variable s is proved but then it is required to show that the RM is asymptotically stable during sliding mode and also to derive the matrix H that will ensure the asymptotic stability of the control system. To show that the RM, as



given in Eq. (61), Eq. (67) is solved to z_2 when $\dot{s} = 0$, to get $z_2 = -Gz_1$. Then sub z_2 in the first equation in Eq. (61) to obtain the dynamics of z_1 during sliding motion

$$\dot{z}_1 = \hat{A}_{r11}z_1 + \hat{A}_{r12}(-Gz_1) + \hat{A}_{R1}x_R = (\hat{A}_{r11} - \hat{A}_{r12}G)z_1 + \hat{A}_{R1}x_R$$

For a controllable pair $(\hat{A}_{r11}, \hat{A}_{r12})$ the matrix G can be selected such that the matrix $(\hat{A}_{r11} - \hat{A}_{r12}G)$ is Hurwitz, which also determine the sliding motion dynamics. Now the objective is to derive the matrix H during sliding mode based on the equivalent control as follow;

$$[\dot{s}]_{eq} = 0 = \hat{A}_{r21}z_1 + \hat{A}_{r22}z_2 + \hat{A}_{R2}x_R + B_{12}u_{eq} + G\hat{A}_{r11}z_1 + G\hat{A}_{r12}z_2 + G\hat{A}_{R1}x_R$$

Solving for u_{eq}

$$u_{eq} = -B_{12}^{-1}\{\hat{A}_{r21}z_1 + \hat{A}_{r22}z_2 + \hat{A}_{R2}x_R + G\hat{A}_{r11}z_1 + G\hat{A}_{r12}z_2 + G\hat{A}_{R1}x_R\}$$

$$= -B_{12}^{-1}\{(\hat{A}_{r21} + G\hat{A}_{r11})z_1 + (\hat{A}_{r22} + G\hat{A}_{r12})z_2 + (\hat{A}_{R2} + G\hat{A}_{R1})x_R\}$$

$$= -B_{12}^{-1}\begin{bmatrix} (\hat{A}_{r21} + G\hat{A}_{r11}) & (\hat{A}_{r22} + G\hat{A}_{r12}) & (\hat{A}_{R2} + G\hat{A}_{R1}) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ x_R \end{bmatrix}$$

$$= -B_{12}^{-1}\begin{bmatrix} (\hat{A}_{r21} + G\hat{A}_{r11}) & (\hat{A}_{r22} + G\hat{A}_{r12}) & (\hat{A}_{R2} + G\hat{A}_{R1}) \end{bmatrix} \hat{x}$$

$$= -B_{12}^{-1}\begin{bmatrix} (\hat{A}_{r21} + G\hat{A}_{r11}) & (\hat{A}_{r22} + G\hat{A}_{r12}) & (\hat{A}_{R2} + G\hat{A}_{R1}) \end{bmatrix} T x_o$$

$\rightarrow u_{eq} = Hx$ where

$$H = -B_{12}^{-1}\begin{bmatrix} (\hat{A}_{r21} + G\hat{A}_{r11}) & (\hat{A}_{r22} + G\hat{A}_{r12}) & (\hat{A}_{R2} + G\hat{A}_{R1}) \end{bmatrix} T$$

Now sub u_{eq} in Eq. (46) to get

$$\dot{x} = Ax + BHx = (A + BH)x$$

For the control system to be asymptotically stable the matrix $(A + BH)$ must be Hurwitz. But since the state is constrained to the sliding manifold $s = 0$ during sliding motion ($s = 0$ is a hyperplane of dimension $n - 1$ in the state space) so one of the eigenvalue of $(A + BH)$ is equal to zero. Accordingly, the remaining $n - 1$ eigenvalues must be of negative real part. Finally, the performance of the proposed SMC to suppress the smart material vibration is determined by the eigenvalues of the control system during sliding motion. If the minimum (but differ from zero) absolute real term of the i^{th} eigenvalue of $(A + BH)$ is greater than that for the original system A , then the control system will attenuate and suppress the smart cantilever vibration effectively. This idea is coined in condition (72).

The sliding mode controller, as in Eq. (66), that granted asymptotic stability of the reduce model, may also cause the instability for the system dynamic which named the control spillover. In the following section the avoideness of spillover problem is proved to happen if the SMC staisfies the performance condition as given in proposition (2).

4.1 Control Spillover Problem and control performance condition

To control vibration of a smart structure, a controller is usually designed based on a reduced order model of the system. When such a reduce order model based controller is applied to the full order system model, the actuating force that reduce the vibration of the lower modes will also influence the residual system model of the structure. Consequently it may produce undesirable vibration due to the unmodeled dynamics. This phenomenon is known as control spillover, **Meirovitch, 1990**.



In the proposed SMC presented in proposition 2, the control spillover is avoided via the second condition imposed on the selection of the matrix G . Moreover the smart cantilver beam dynamics with the SMC will be given by;

$$\dot{x} = Ax + BHx = (A + BH)x \quad (73)$$

which represent the whole model matrix after applying the sliding mode control.

Remark (3): the smart cantilver beam dynamics as given in Eq. (73) is derived based on the equivalent control concept in SMC theory.

Remark (4): The second condition ii. can be used in measuring the performance of the proposed controller; where a higher vibration suppression can be achieved for large difference between R^c and R^o .

5. SIMULATION RESULTS AND DISCUSSION

In this section the simulation results for a smart cantilever beam, which is subjected to an initial tip deflection, are presented. MATLAB software is used as a simulator to the cantilever beam system. The physical and geometrical specifications for the beam are given in **Table 1** below. To show that the derived model represents the system dynamics at least with respect to the dominant natural frequencies, the natural frequencies of the beam (Eq. 47) are calculated and compared with the natural frequencies obtained from the ANSYS program. The results are shown in **Table 2** with a good agreement.

The balance realization and order reduction process for the system model had been performed to reduce its states form (24) states to (3) states, without significant affect to its dominant mode. This is demonstrated in **Fig.2** in the Bode plot. The number of states is equal to the selected singular values in **Table 3** for the diagonal elements of matrix Σ (the diagonal elements are the singular values of the grammians σ_i , $i = 1, 2, \dots, n$). Accordingly the reduced order model matrices are determined according to section three with appropriate dimension and only three states. By using the reduce order model states $x_r \in \mathcal{R}^3$, the designed sliding mode controller is applied to the cantilever beam and the system is simulated for 15 mm initial tip displacement. To investigate the stability of the smart cantilever, (the total system model with the SMC), the eigenvalues are determined based on the equivalent control, i.e., during sliding motion. The new system eigenvalues are presented in the second column of **Table 4**. In this table, in the second column, one of the eigenvalues (before the last one) is nearly equal to zero, while the minimum (but differ from zero) absolute real term is greater than that for the original system A (the first column). This agrees with what has been pointed in section four.

For the first set of numerical simulation, a (0.00001) second is used as a period of integration to the sliding mode control system. In **Fig.3**, the controlled tip displacement is compared with the open loop case. The ability of the SMC in stabilizing the tip displacement is clarified in this figure where it required about 2.5 second only. In addition, the control input voltage to the piezoelectric element, shown in **Fig.4**, is switched between 200 and -200 volt. This is a consequence of the discontinuous nature of the proposed SMC. Additionally the sliding variable is plotted in **Fig.5** where it reaches zero value after a very small period of time. The oscillation of the sliding variable is because that the sliding variable dynamics is affected also by the remaining states which ignored during getting the reduced order model.



In the real application of the suggested controller, it may be difficult to use 0.00001 second as sampling time for the control input where it is required to change the control voltage after each 0.00001 second. In order to access the real situation, the time period for the control input is taken equal to 0.0025 second. Consequently, the second set of numerical simulation use these time numerical values. The control performance is similar to the first set of simulation as clarified in **Figs.6, 7 and 8** for the sliding variable, the tip displacement and the control input voltage respectively where, as can be seen, the vibration suppression ability is nearly the same as in the first set of simulation. This enhances the applicability of the suggested controller.

From **Figs.4 and 8** it can be seen that the control input voltage to the piezoelectric still actuated in full value in spite of the sliding variable equals zero approximately. This makes system chatter because the control voltage switches between the full input voltage due to the oscillation of the sliding variable s around the zero with a very small amplitude. To remove or attenuate the chattering effect, the *signum* function Eq. (69) is replaced by a continuous approximate function (the arc tan function), **Edwards, et al., 1998**. It is given by;

$sgn(s) \approx \frac{2}{\pi} * \tan^{-1}(10 * s)$. Replacing $sgn(s)$ by the approximation given above will prevent chattering and smoothing the values of the input voltage as shown in Fig (9). Eventually the sliding variable and the tip displacement are shown in **Figs.10 and 11** respectively, which proves that the control performance is as in the case of using signum function but with control voltage tends to zero after the beam vibration is die out.

6. CONCLUSIONS

In this paper, the sliding mode was designed to suppress the vibration induced in a cantilever smart beam subjected to an initial tip displacement. The state space model is obtained using the finite element approach and modal analysis resulting after appropriate modal reduction. During the theoretical calculations, the 24th order system model obtained from the finite element model is reduced to the three order using a model reduction technique based on balance realization without affecting its dominant modes. For the proposed SMC, the control system stability and the control performance condition are derived in Eq. (71) and inequality (71) respectively. When the $(A + BH)$ matrix has $(n - 1)$ negative roots plus one equal to zero the control system is asymptotically stable and the control spillover is avoided. These results were proved by making the derivative of the candidate Lyapunov function negative definite and using the equivalent control concept and also clarified in **Table 4** where 23 negative real eigenvalues plus one value very close to zero (1.056e-09). In addition, the control performance was maintained at the desired level via satisfying inequality (72) as can be detected in **Table 4** where the largest eigenvalue (has negative sign) for the closed loop system is smaller than that for the open loop case. The numerical simulations prove the effectiveness and performance of the proposed SMC where the cantilever beam vibration is suppressed in effective way when compared to the open loop case. Finally, and in order to overcome the chattering problem the signum function was replaced with the approximation given by the arctan function with appropriate parameters. The chattering is attenuated; the sliding variable and the control voltage input are accordingly smoothed as shown in **Figs.9 to 11** with the same control performance.

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NOMENCLATURE

a_1 to a_4	constants used in solution of the displacement function, dimensionless.
A_b	cross-section area of the beam element, mm^2 .
A_p	cross-section area of the piezoelectric element, mm^2 .
A	state matrix.



b	width of the beam, mm.
B	input matrix.
c	constant which equal to $\sqrt{EI/\rho A}$
C	output matrix.
d_{31}	piezoelectric constant, m/V.
e_{31}	piezoelectric stress/charge constant, VmN^{-1}
E_b	young modulus of the beam, GPa.
E_p	young modulus of the piezoelectric, GPa.
f_{ext}	external force, N.
f_{ctrl}	control force, N.
F_1, F_2	force acting at the node, N
G_c	signal condition device, dimensionless.
h	constant vector.
$i(t)$	sensor current, Amps.
k_b	stiffness matrix of the beam element.
k_p	stiffness matrix of the piezoelectric element.
l_b	length of the beam element, mm.
L	length of beam, mm.
M_b	mass matrix of the beam element.
M_p	mass matrix of the piezoelectric element.
N	shape function.
q	vector displacement.
\dot{q}	velocity vector.
\ddot{q}	acceleration vector.
t_a	thickness of the actuator, mm.
t_b	thickness of the beam, mm.
T	kinetic energy.
u	control input, Volt.
U	Strain enegy.
V_a	actuaor voltage, Volt.
V_s	sensor voltage, Volt.
$w(x, t)$	displacement function.
w	dgree of freedom.
α, β	damping coefficient, dimensionless.
ε	strain.
ρ_b	density of the beam, kg/m^3
ρ_p	Density of the piezoelectric patch kg/m^3

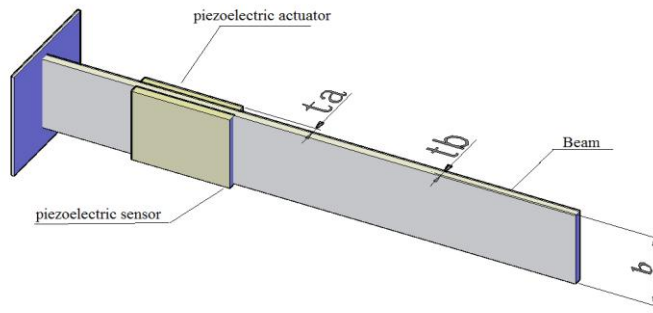


Figure 1. Clamped-free flexible smart beam model

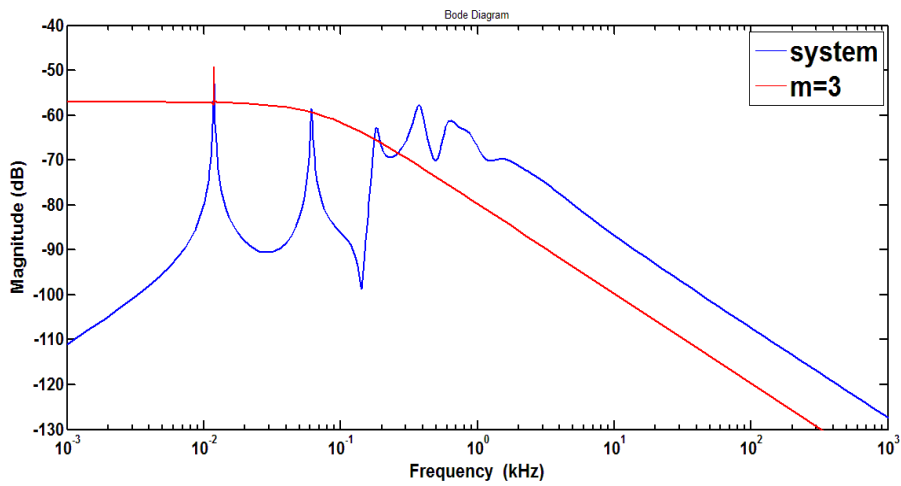


Figure 2. Bode plot

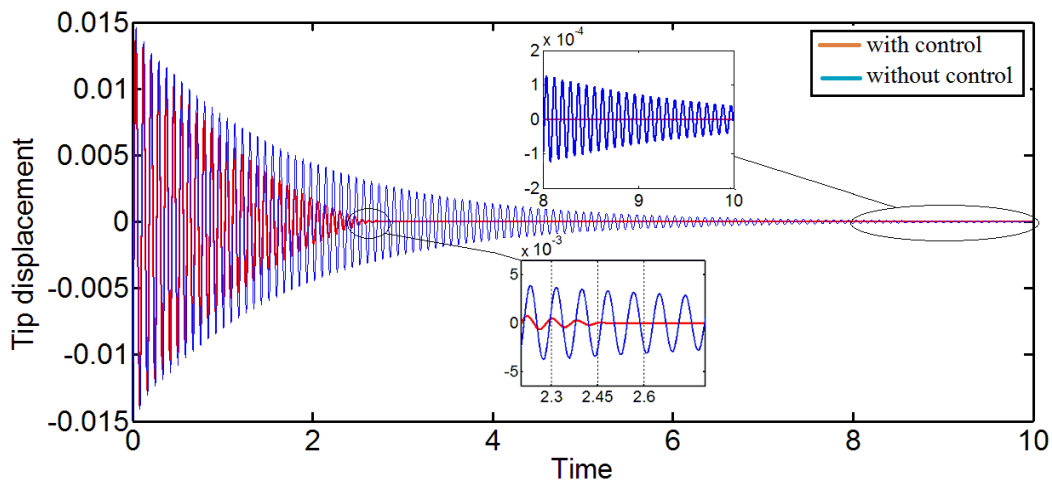


Figure 3. Tip Displacement for open loop and closed loop control system

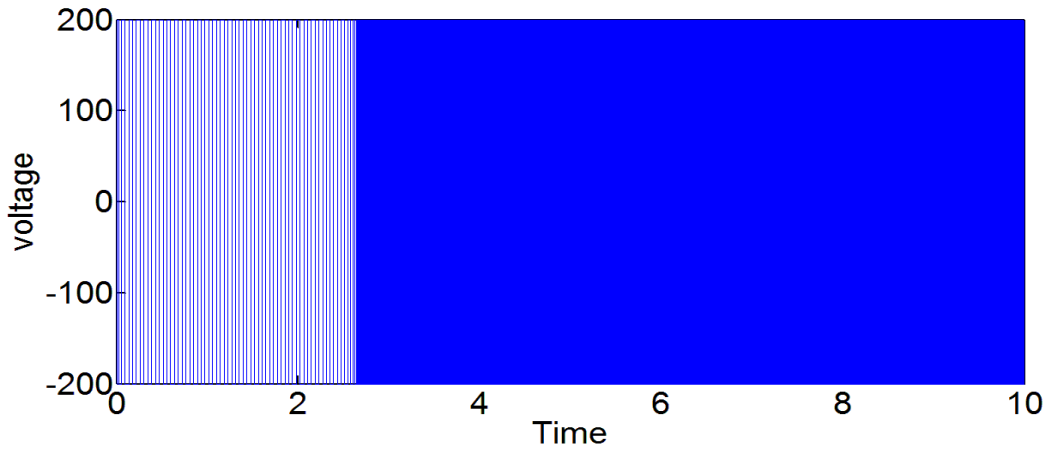


Figure 4. The control input voltage to the piezoelectric

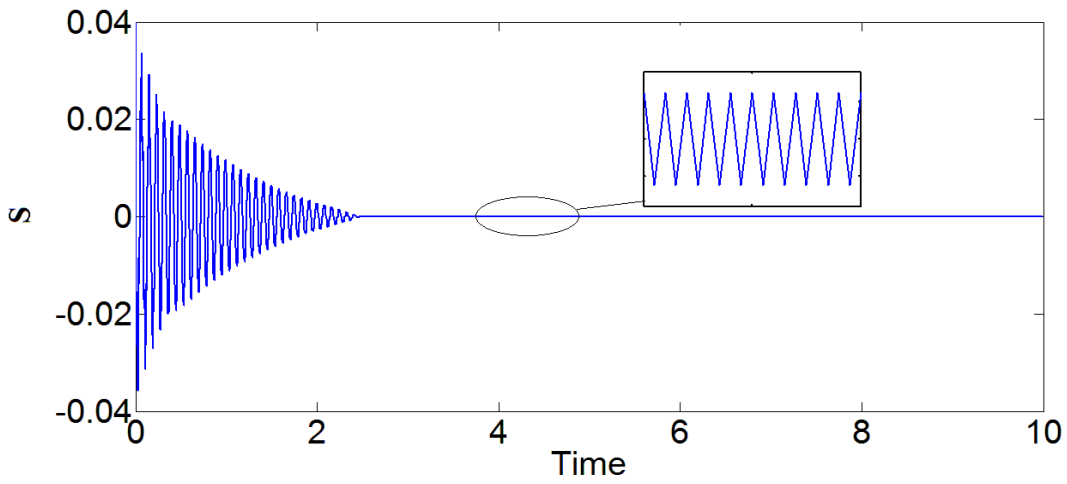


Figure 5. The sliding variable s

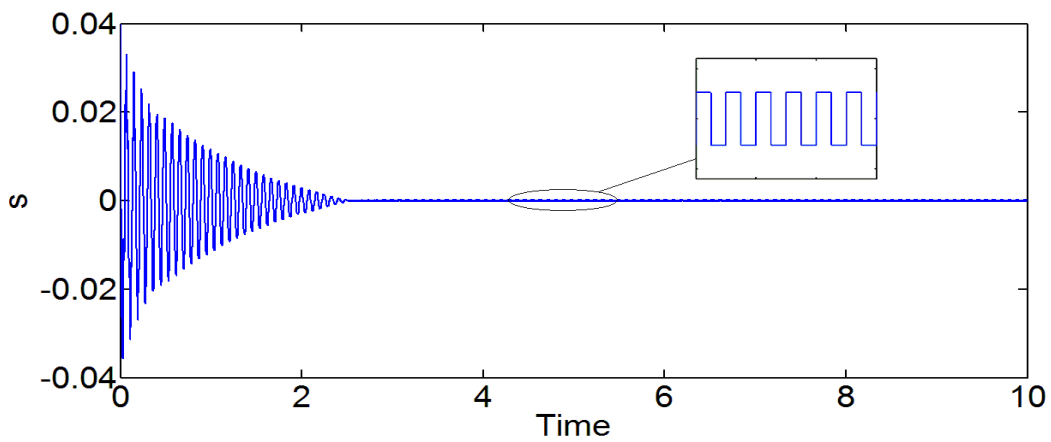


Figure 6. The sliding variable s

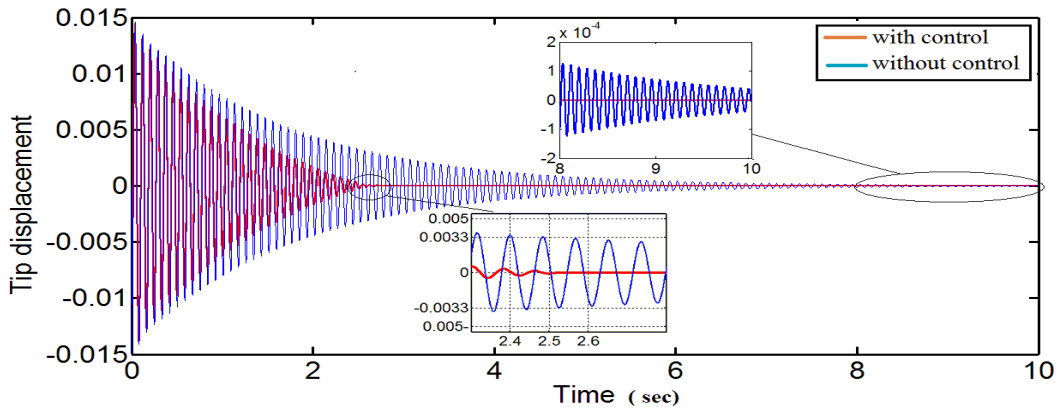


Figure 7. Tip Displacement for open loop and closed loop control system

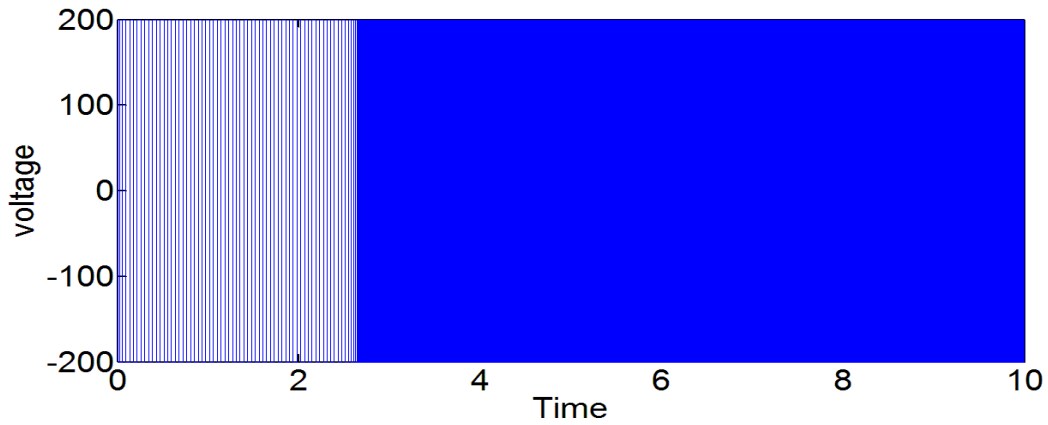


Figure 8. The control input voltage to the piezoelectric

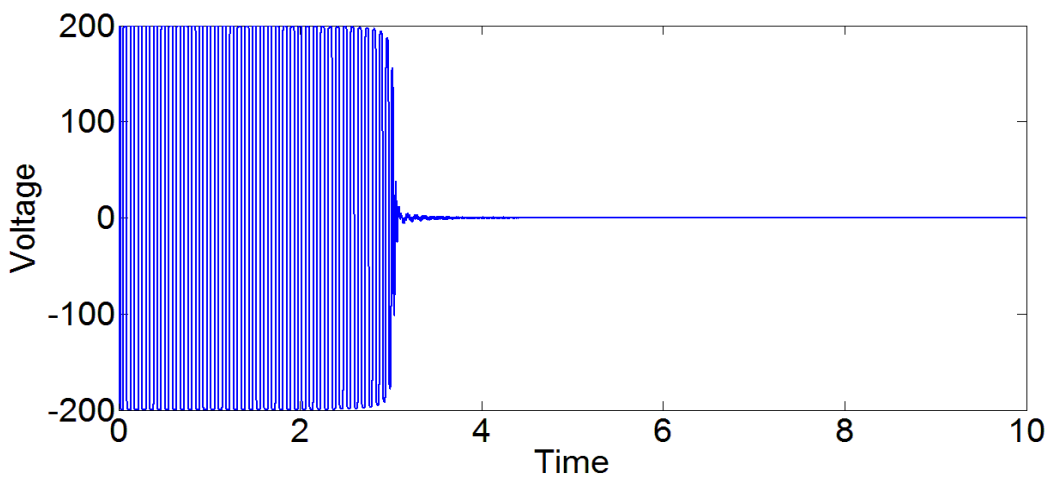


Figure 9. Control input voltage to the piezoelectric by using approximate function

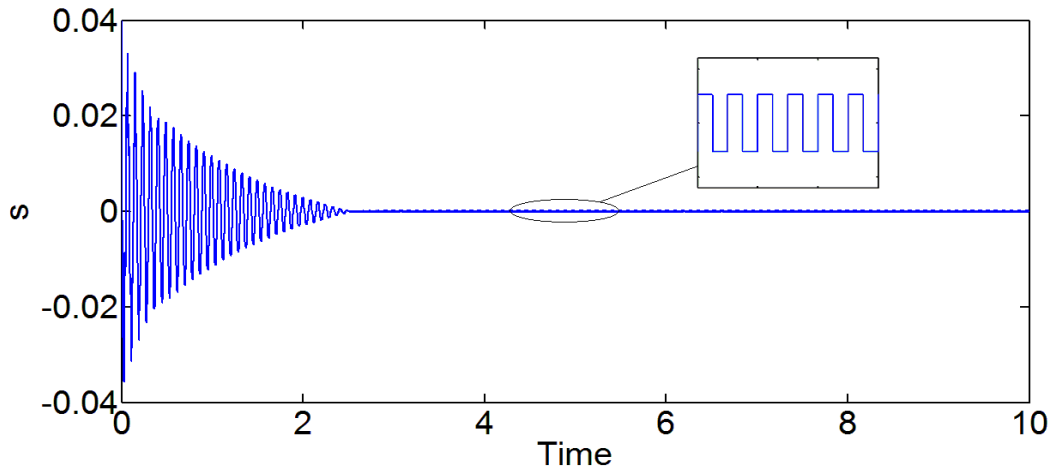


Figure 10. The sliding variable s using approximate function

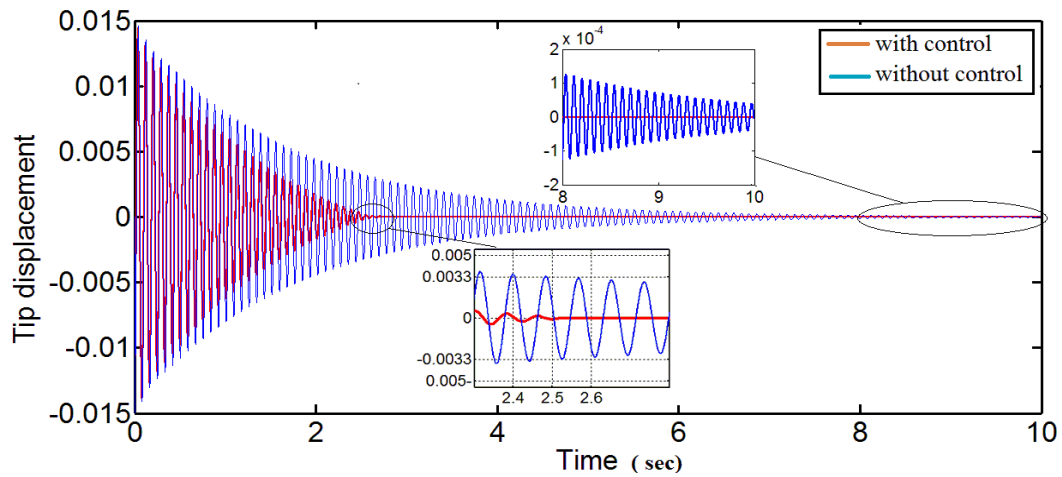


Figure 11. Tip Displacement for open loop and closed loop control system using approximate function

Table 1. The specification for the flexible cantilever beam and piezoelectric

Physical Specification	Cantilever Beam (st-st)	Piezoelectric
Length	$L=276$ mm	$l_a =46$ mm
Width	$b =33$ mm	$b= 33$ mm
Thickness	$t_b =1$ mm	$t_a = 0.762$ mm
Young modulus	$E_b =193.06$ Gpa	$E_p = 68$ Gpa
Density	$\rho_b =8030$ Kg/m ³	$\rho_p = 7700$ Kg/m ³
Damping coefficients	$\alpha = 0.8$ & $\beta = 6.8E-5$	



Table 2. Natural frequency results of the system

Natural Frequency	MATLAB (Hz)	ANSYS (Hz)	Error%
f_1	11.878	11.421	3.125
f_2	61.376	61.148	0.372
f_3	181.06	180.1	0.558
f_4	153.5	151.3	1.45

Table 3. Singular value of grammians matrix

$\Sigma =$	$diag \{ 0.0011404 \quad 0.0011399 \quad 0.000698 \quad 0.0006932 \quad 0.00058469$ $0.00057315 \quad 0.00040318 \quad 0.00037605 \quad 0.00034589 \quad 0.00032743$ $0.00018763 \quad 9.8742e - 05 \quad 6.1004e - 05 \quad 5.7566e - 05 \quad \dots \dots \}$
Selected singular values for model reduction	$\sigma_{(1)} = dig\{0.0011404 \quad 0.0011399 \quad 0.000698\}$



Table 4. System eigenvalues and controlled system eigenvalues

System Eigenvalues	Controlled System Eigenvalues
-96730 + 0i	-96757 + 0i
-84999 + 0i	-85330 + 0i
-21374 + 13107i	-20079 + 14749i
-21374 - 13107i	-20079 - 14749i
-11184 + 14277i	-10038 + 17969i
-11184 - 14277i	-10038 - 17969i
-17782 + 0i	-17721 + 0i
-17342 + 0i	-17338 + 0i
-6491.8 + 12198i	-6289.6 + 11276i
-6491.8 - 12198i	-6289.6 - 11276i
-3092.9 + 9021.6i	-4558.6 + 7464.2i
-3092.9 - 9021.6i	-4558.6 - 7464.2i
-1122.7 + 5634.6i	-1898.7 + 5931.3i
-1122.7 - 5634.6i	-1898.7 - 5931.3i
-482.27 + 3733.7i	-285.41 + 3354.9i
-482.27 - 3733.7i	-285.41 - 3354.9i
-190.95 + 2359.6i	-135.03 + 1341.8i
-190.95 - 2359.6i	-135.03 - 1341.8i
-44.402 + 1136.8i	-3.3486 + 1148.7i
-44.402 - 1136.8i	-3.3486 - 1148.7i
-5.4563 + 385.6i	-14.623 + 386.09i
-5.4563 - 385.6i	-14.623 - 386.09i
-0.58936 + 74.626i	1.056e-09 + 0i
-0.58936 - 74.626i	-43.536 + 0i