



SDPLL-Based Frequency Estimation of a Sinusoid in Colored Noise

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ABSTRACT:

The problem of frequency estimation of a single sinusoid observed in colored noise is addressed. Our estimator is based on the operation of the sinusoidal digital phase-locked loop (SDPLL) which carries the frequency information in its phase error after the noisy sinusoid has been acquired by the SDPLL. We show by computer simulations that this frequency estimator beats the Cramer-Rao bound (CRB) on the frequency error variance for moderate and high SNRs when the colored noise has a general low-pass filtered (LPF) characteristic, thereby outperforming, in terms of frequency error variance, several existing techniques some of which are, in addition, computationally demanding. Moreover, the present approach generalizes on existing work that addresses different methods of sinusoid frequency estimation involving specific colored noise models such as the moving average (MA) noise model. An insightful theoretical analysis is presented to support the practical findings.

Keywords: Sinusoidal digital phase-locked loop (SDPLL); Cramer-Rao bound (CRB); colored noise; frequency error variance; signal-to-noise ratio (SNR).

تقدير التردد للموجة الجيبية في الضوضاء الملونة اعتماداً على حلقة اقفال الطور الرقمية الجيبية

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الخلاصة:

هذا البحث يتناول مشكلة تقدير التردد للموجة الجيبية المفردة في وجود الضوضاء الملونة. ان المقدر المتبع يركز على عمل حلقة اقفال الطور الرقمية الجيبية التي تحمل معلومات التردد داخل خطأ الطور بعد الحصول على الموجة الجيبية المشوبة بالضوضاء بواسطة حلقة اقفال الطور. وضحنا عن طريق برنامج محاكاة بالحاسبة ان مقدر التردد يتغلب على الحد الأدنى لمحدد كيريمر في خطأ التغير لنسب الإشارة-إلى-الضوضاء المتوسطة والعالية عندما تكون خصائص الضوضاء الملونة تشبه الخصائص العامة لمرشح الترددات الوطنية، وبذلك يتحسن الاداء، مقاساً بصيغة خطأ التغير بالتردد، عن عدة تقنيات موجودة والتي يتطلب قسم منها عباً حسابياً عالياً. إضافة إلى ذلك فان الطريقة الحالية يمكن اعتبارها تعميماً للبحوث الحالية التي تتناول عدة طرق لتقدير التردد الجيبية تتضمن نماذج معينة للضوضاء الملونة مثل نموذج المعدل المتحرك للضوضاء. وتم تقديم تحليل نظري حدسي لدعم النتائج العملية.

الكلمات الرئيسية: حلقة اقفال الطور الجيبية الرقمية؛ محدد كيريمر؛ الضوضاء الملونة؛ خطأ التغير بالتردد؛ نسبة الإشارة – إلى – الضوضاء.

1. INTRODUCTION

Rapid frequency estimation of a sinusoid in the presence of noise is a problem that is frequently encountered in signal processing and communications with applications varying from radar, sonar, signal interception and detection, carrier synchronization and many others. Many methods of sinusoidal frequency estimation have been developed especially for the white noise case. The maximum likelihood (ML) estimator involving the location of the peak of the periodogram is well-known. The performance of the ML estimator in terms of estimator error variance achieves the CRB at high signal-to-noise ratios (SNR's), but at the expense of a large computational complexity, even when the fast Fourier transform (FFT) is used (Fu H. et al, 2007). The ML estimator is also a batch processing technique starting the processing only after all samples have been received (Richard Brown III D. et al, 2010). Subsequent to the introduction of the ML method, several fast and accurate sinusoidal frequency estimators in the presence of white noise have been reported (Kay S., 1989) attaining the CRB on variance for high enough SNR.

Frequency estimation techniques driven by a frequency tracking loop such as the PLL have also been reported in the literature in the context of operation in a white noise environment (Sithampanathan K., 2008). The advantage of such an approach is that the PLL acts as a dynamic band-pass filter (BPF) to track the frequency, which improves the SNR due to reduced bandwidth (BW) once the signal is acquired. This improvement in SNR certainly leads to an improvement in the frequency error variance for a given system. Indeed, it has been shown that such a frequency estimator using an arc-tan DPLL (Sithampanathan K., 2008) beats the CRB for low numbers of estimation samples. The appeal of DPLLs is also evident in that they offer low-complexity sample-by-sample operation suitable for real-time applications (Richard Brown III D. et al, 2010).

In this paper, we improve on the work in (Sithampanathan K., 2008) to include the case where the accompanying noise is colored. This case occurs in many applications and is definitely more practically relevant (Stoica P. et

al, 1997) In our work, however, we choose to investigate the SDPLL as a frequency estimator in colored noise. Among DPLL's, the uniform sampling SDPLL's are particularly popular, as they are simple to implement and suitable for relatively wide locking ranges (Hussain Z. M. et al, 2011). (Elasmi-Ksibi R. et al, 2010) have recently treated the case of frequency estimation of a sinusoid in colored noise exploiting high-order lags of the autocorrelation function (ACF) of the noisy sinusoid, and consequently restricting the investigation to MA noise models but achieving low computational complexity. The present SDPLL-based work compares favorably with (Elasmi-Ksibi R. et al, 2010) as regards computational complexity, the extension of the MA case to generally any LPF characteristic, in addition to attaining and even surpassing the CRB associated with the white noise case for a wide range of SNR values extending from moderate to high values. A high-pass filtered (HPF) characteristic of noise, however, is shown to render the system incapable of attaining the CRB. Obviously, however, long settling time and the presence of overshoot are drawbacks of PLL frequency estimation (Saber M. et al, 2011).

The rest of the paper is organized as follows: Section 2 summarizes the SDPLL analysis and explains the design methodology. Sections 3 and 4 focus on the SDPLL as a frequency estimator in white and colored noise respectively, analytically highlighting the improvement on the error variance that is achieved with a low-pass filtered noise characteristic. Section 5 presents simulation results of the proposed system. Finally, Section 6 concludes the paper.

2. ANALYSIS AND DESIGN OF THE NOISE-FREE SDPLL

SDPLLs are non-uniform sampling sinusoidal PLLs that have the advantage of being simple to implement and having relatively wide locking ranges. The block diagram of this system is shown in **Fig. 1**. The SDPLL consists of a sampler-ADC unit which serves as a phase detector, a digital LPF (DF), and a digital voltage-controlled oscillator (DCO). The latter provides constant-amplitude but variable-frequency output pulses that control the sampling instants of the sampler-ADC unit (Hussain Z. M. et al, 2011).



The DF output is input to the DCO to vary its phase and hence sampling instant of the input analog signal $x(t)$. When the signal is absent, the DCO runs at its free-running frequency which we call f_o . When the sampler takes a sample of $x(t)$ at the k th sampling instant, the ADC converts the analog value of the input into a digital value that the DF uses at its input to yield $y(k)$ at its output. This digital $y(k)$ is then input to the DCO. As the DCO input changes, so does the sampling period $T(k)$ of the ADC. The sampling process is non-uniform, i.e. the sampling frequency is not constant. Under certain conditions, namely, the locking conditions, the convergence or locking occurs such that $T(k)$ approaches the inverse of f_i , the input sinusoid frequency. **Fig. 2** shows the waveforms associated with the SDPLL.

For the dual purpose of clarity and symbols unification, we find it convenient to summarize the elementary analysis of the SDPLL already present in the literature such as in (Hussain Z. M. et al, 2011).

Assume the sinusoidal input is given by:

$$x(t) = A \sin(\omega_i t + \theta_o) \quad (1)$$

where A is the signal amplitude, $\omega_i = 2\pi f_i$ is the sinusoidal angular frequency, and θ_o is a constant phase. The locking range of the loop is the range of instantaneous frequencies that the loop can track. Since the locking range is dependent on the deviation of ω from the loop center frequency ω_o , we can then write eq. (1) as:

$$x(t) = A \sin[\omega_o t + \theta(t)] \quad (2)$$

$\theta(t)$ is the information-bearing phase given by:

$$\theta(t) = (\omega_i - \omega_o)t + \theta_o = \Delta\omega t + \theta_o$$

After sampling, the input signal at the k th sampling instant $t(k)$ will take on the following form:

$$x(k) = A \sin[\omega_o t(k) + \theta(k)] \quad (3)$$

Now we can define the input phase at the k th sampling instant as:

$$\phi(k) = \omega_o t(k) + \theta(k)$$

The sampling interval of the DCO at the k th sampling instant is given by:

$$T(k) = T_o - y(k - 1) \quad (4)$$

T_o is the DCO free-running period. This shows that the k th output sample of the DF, $y(k)$, effectively determines the sampling period $T(k+1)$. Eq. (4) reveals how the DCO operates; it decreases its period (increases its frequency) as the DF output increases.

From the above findings, the phase difference equation of the system is given by (Hussain Z. M. et al, 2011):

$$\phi(k + 1) - \phi(k) = \theta(k + 1) - \theta(k) + \omega_o y(k)$$

Let us assume that the digital filter transfer function is $H(z)=G_1$ where G_1 is a constant. This yields a first-order SDPLL. Then, the above equation is used to modify the difference equation to:

$$\phi(k + 1) = \phi(k) - K_2 \sin[\phi(k)] + \Lambda_o$$

where $\Lambda_o = 2\pi(\omega_i - \omega_o) / \omega_o$ and $K_2 = \omega_i G_1 A$.

Defining K_1 to be $K_1 = \omega_o G_1 A$ and W as the frequency ratio $W = \omega_o / \omega_i$, then:

$$K_2 = K_1(\omega_i / \omega_o) = K_1 / W$$

The parameters K_1 and W control the locking range. The locking conditions have been derived in (Hussain Z. M. et al, 2011). We state here the results of the derivation in (Hussain Z. M. et al, 2011) as these results are key attributes in SDPLL design.

$$\begin{aligned} K_1 &> 2\pi|1-W| \\ K_1 &< \sqrt{(4+4\pi^2)W^2 - 8\pi^2W + 4\pi^2} \end{aligned} \quad (5)$$

The above two equations specify a range of K_1 that ensures that the input frequency is within the frequency locking range of the first-order SDPLL.

According to the above inequalities, we may design our first-order SDPLL. For example, if we choose a DCO with $f_o = 1\text{Hz}$ and the input frequency $f_i = 0.83\text{Hz}$, then clearly $W=1.2$. And from the locking conditions of eq. (5), K_1 can be safely chosen as $K_1=1.7$.

3. THE SDPLL AS A FREQUENCY ESTIMATOR OF A NOISY SINUSOID

The SDPLL can detect the input frequency in the presence of noise, even for low SNRs. Having designed the first-order SDPLL in Section II, we now explain the frequency estimator driven by the SDPLL when the input sinusoid is contaminated by additive white Gaussian noise (AWGN). The noisy input sinusoid may be written as:

$$x(k) = A \sin[\phi(k)] + n(k) \quad (6)$$

where $n(k)$ is additive noise. We are interested in estimating the frequency of the noise-free input sinusoid which we call f_i . The SDPLL is a computationally efficient sample-by-sample method to extract f_i from noisy observations. The PLL acts as a dynamic BPF that tracks the input frequency improving the SNR by bandwidth reduction. The SNR improvement leads to an improvement in frequency error variance. Using the symbols in Section 2, the sequence $t(k)$ can be used to generate a frequency sequence $f_n(k)$ by inverting the interval between each two consecutive samples of $t(k)$ and assigning it to $f_n(k)$. The probability density function of the instantaneous frequency $pdf(f_n)$ has a maximum at approximately $f = f_i$. The variance of this frequency estimator is a function of the SNR; it decreases as the SNR increases. **Fig. 3** shows

$pdf(f_n)$ for the design of Section 2 and for the white noise case.

4. THE SDPLL FREQUENCY ESTIMATOR OF A SINUSOID IN COLORED NOISE

In this paper, we wish to investigate the colored noise case. In (Elasmi-Ksibi R. et al, 2010), a novel method of frequency estimation of a sinusoid in colored noise using multiple autocorrelation lags is presented. However, the method is applicable only to environments where the colored noise is of finite memory, i.e. moving average (MA) noise models. The method features low computational complexity and compares favorably with its forerunners. We show that frequency estimation in colored noise is possible with different types of low-pass-filtered noise, including the MA noise model, whereas the method in (Elasmi-Ksibi R. et al, 2010) is applicable only to the MA model.

In particular, for the SDPLL, we find that low-pass-filtered colored noise, added to the input sinusoid whose frequency is to be estimated, significantly enhance the SNR when compared to the white noise case, whereas a high-pass-filtered noise has the adverse effect. We have also found that this is true for any kind of low-pass-filtered noise. We simulate the colored noise by passing AWGN through a first-order IIR LPF first, then a FIR LPF of a MA type. We can explain this favorable behavior of the SDPLL as follows: Since the output pulses from the DCO control the sampling of the sine-plus-colored-noise signal, the digital LPF sampling frequency has, on average, a value of $f_i = 0.83$ Hz according to the design of Section 2. Therefore, the folding frequency (corresponding to the digital frequency of π radians) is $f_i/2 = 0.415$ Hz. The periodic frequency transfer function of this digital LPF has a typical form shown in **Fig. 4**. It is clear that this filter has high gain around 0.83 Hz. This means that LPF colored noise boosts the SNR thereby improving the frequency error variance for the SDPLL acting as a frequency estimator. On the other hand, a HPF for producing colored noise has the typical transfer function shown in **Fig 5**. The presence of a peak at half the input frequency, i.e. at 0.415 Hz, results in a degradation of the



performance of the SDPLL as a frequency estimator and the frequency error variance increases.

As in any estimation problem, the idea is to produce from a noisy observation vector f_n an unbiased estimate \hat{f}_i of a deterministic parameter f_i . The estimation error variance is lower-bounded by the Cramer-Rao bound (CRB):

$$E[(\hat{f}_i - f_i)^2] \geq CRB \quad (7)$$

The CRB is given by:

$$CRB = E \left[\frac{d}{df_i} \ln(p(f_n; f_i)) \right]^2 \quad (8)$$

The probability density function $p(f_n; f_i)$ of f_n , corresponding to a given value of f_i , is called the likelihood function of f_i , while $\ln(p(f_n; f_i))$ is the log-likelihood function of f_i . The expectation is with respect to $p(f_n; f_i)$.

The CRB in our work is the lower bound of the frequency error variance as in eq. (7). Analytical evaluation of the CRB (Vaseghi S.V., 2006) is usually extremely difficult, if not impossible; therefore we resort to computer simulation to find it (Noels N. et al, 2002).

5. SIMULATION RESULTS

The first-order SDPLL is simulated in MATLAB 7. The program for the white noise case is listed below in pseudo-code; the symbols are as presented in Section 2.

SDPLL algorithm for sinusoid frequency estimation

INPUT: Noise vector n , and parameters from which to compute sinusoid vector x , plus other loop parameters.

OUTPUT: Frequency error variance vector vs. SNR vector.

*/*Loop parameters*/*
Frequency ratio (W)=1.2

Design Constant Value (K1)=1.7

Input phase angle (θ_o)=1 rad

*/*Other parameters*/*

Sinusoid amplitude (A)=1

Center frequency (fo)=1 Hz

Center angular frequency (wo)=2*pi*fo rad/s

DCO period (To)=1/fo s

Input sinusoid frequency (fi)=fo/W Hz

Input sinusoid angular frequency (wi)=2*pi*fi rad/s

Constant (K2)=K1/W

Digital filter gain (G1)=K1/(wo*A)

*/*Main loop*/*

FOR Signal/Noise ratio (SNR)=0 to 30 step 10 **DO**

Standard deviation (sd)=sqrt(0.5/10^(SNR/10))

*/*Initialization*/*

First noisy sine sample (xo)=A*sin(θ_o)

+sd*n(1)

Filter output (yo)=G1*xo

First period T(1)=To-yo

First sampling instant t(1)=T(1)

*/*For the rest of the samples*/*

FOR k=1 to 999 **DO**

Noisy sine sample x(k)=A*sin(wi*t(k)+

θ_o)+sd*n(k+1)

Filter output y(k+1)=G1*x(k)

Sampling period T(k+1)=To-y(k)

Sampling instant t(k+1)=t(k)+T(k+1)

END FOR

Frequency vector fn=1/T

Frequency variance=variance(fn)

END FOR

RETURN dB-variance vs. SNR

The corresponding MATLAB code yields the frequency error variance versus SNR. For the colored noise case, we pass the AWGN through a LPF transfer function to produce colored noise. We demonstrate two cases:

Case I: IIR first-order LPF. The transfer function is given by:

$$H(z) = \frac{z}{z - a}$$

If we assume that σ_i^2 is the input noise power to the LPF, then it is straightforward to prove that:

$$\sigma_o^2 = \frac{\sigma_i^2}{1-a^2}$$
 is the output noise power.

We assume that the input noise power (variance) is equal to unity. Therefore, to compare between white and colored noise effects, we have to equate the powers of the two. Thus, we multiply the colored noise sequence by $\sqrt{1-a^2}$ before adding it to the sine wave. If we take $a=0.9$, then we have to multiply by $\sqrt{0.19}$. After making the necessary modification to the above pseudo-code and the corresponding MATLAB code, we plot the variance versus SNR on the same graph as that for the white noise case as in **Fig. 6**. The variance versus SNR for the corresponding HPF filtered colored noise case is also shown in the figure.

Case II: We also test low-pass filtered noise using a MA process. We choose the FIR MA filter transfer function as:

$$H(z) = 0.2(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}).$$

The power relation is:

$$\sigma_o^2 = \sigma_i^2 \sum_{l=0}^4 b_l^2 = \sigma_i^2 / 5.$$

The b_l 's are the impulse response values of the FIR MA filter.

Also assuming unity input noise power, we have to multiply the filter output by $\sqrt{5}$ for a meaningful comparison. The resulting curve is also included in **Fig. 6**, and is shown to coincide nearly with IIR filter case. The conclusion is that for LPFs, in general, the frequency error variance is improved (decreased).

The Cramer Rao bound (CRB) is a lower bound on the error variance of any unbiased estimator and serves as an important benchmark to compare practical estimators (Noels N. et al , 2002), (Vaseghi S.V., 2006). **Fig. 6** is extended to include high SNRs. The CRB is the asymptote to the curves as SNR increases. Therefore, we can find it practically as the figure shows. **Fig. 7** is a zoom-in to the moderate-to-high SNR region of **Fig. 6**. It can be seen from **Fig. 7** that when the additive noise

is a low-pass-filtered one, we gain an improvement over the white noise case as regards the variance of the frequency estimate for moderate and high SNRs.

6. CONCLUSIONS

An SDPLL-based single-sinusoid frequency estimator in white and colored noise has been simulated and tested. It has been shown that when the additive noise to the sinusoid has a LPF characteristic we gain an improvement of the estimator performance in terms of reduced frequency error variance regardless of the type of LPF whether IIR or FIR. This work is an extension of related PLL-based methods reported in the literature for the white noise case, and of specifically MA-noise-model non-PLL-based frequency estimators. We have shown through simulation that our estimator beats the CRB of the white noise case for moderate and high SNRs.

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SYMBOLS AND ACRONYMS

A the signal amplitude

b_l 's impulse response values of the FIR MA filter

f_i frequency of the noise-free input sinusoid

\hat{f}_i unbiased estimate of f_i

f_n noisy observation vector

T_o DCO free-running period

T(k) sampling period of the ADC

W frequency ratio

x(t) input analog signal

y(k) input to the DCO

θ_o constant phase

$\theta(t)$ information- bearing phase

σ_i^2 The input noise power to the LPF

$\phi(k)$ input phase at the kth sampling instant

ω_i sinusoidal angular frequency

ω_o loop center frequency

ACF autocorrelation function

ADC analog to digital converter

AWGN additive white Gaussian noise

BPF band-pass filter

BW bandwidth

CRB Cramer-Rao bound

DCO digital voltage-controlled oscillator

DF digital filter

DPLL digital phase-locked loop

FFT fast Fourier transform

FIR finite impulse response

HPF high-pass filter

IIR infinite impulse response

LPF low-pass filter

MA moving average

ML maximum likelihood

pdf probability density function

PLL phase-locked loop

SDPLL sinusoidal digital phase locked loop

SNR signal-to-noise ratio

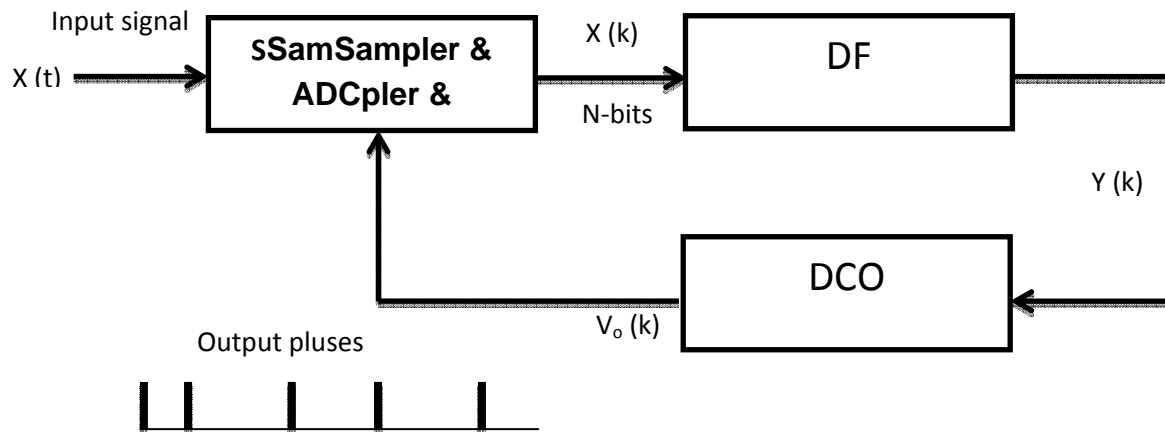


Fig. 1: A block diagram of SDPLL (Hussain Z. M. et al., 2011).

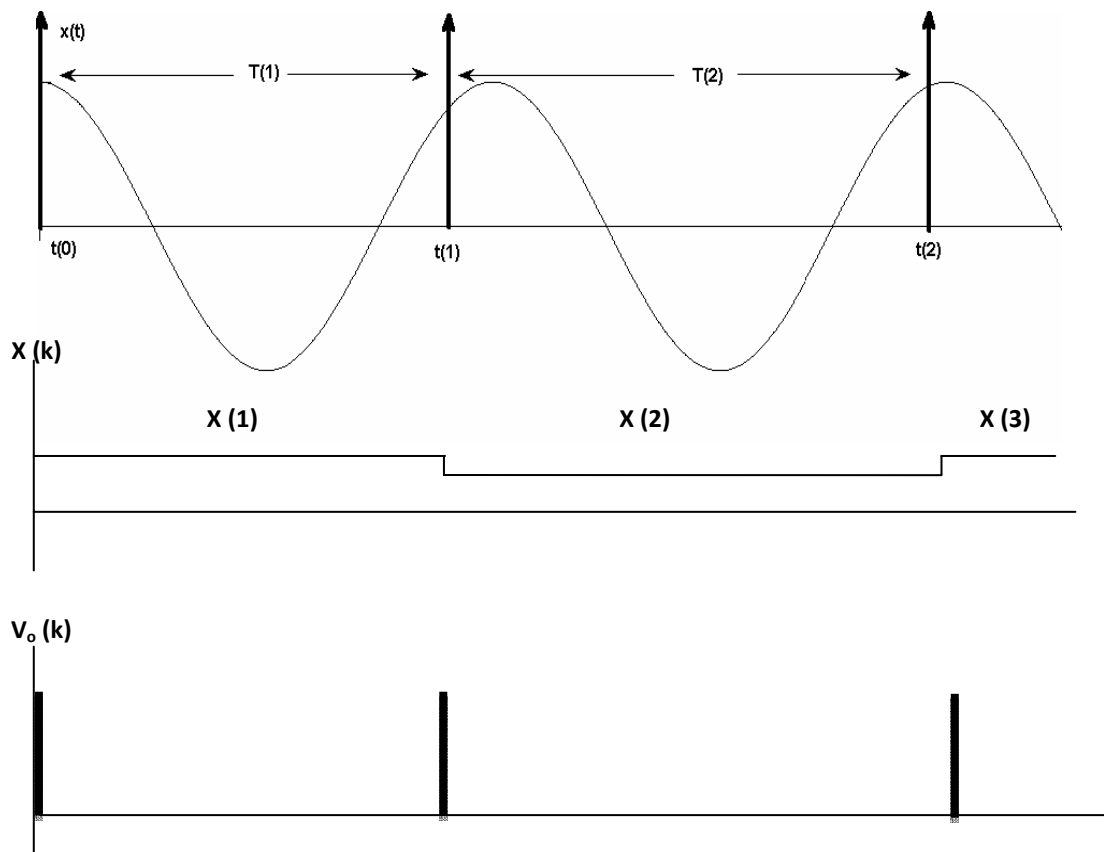


Fig. 2: Waveforms associated with SDPLL (Hussain Z. M. et al., 2011).

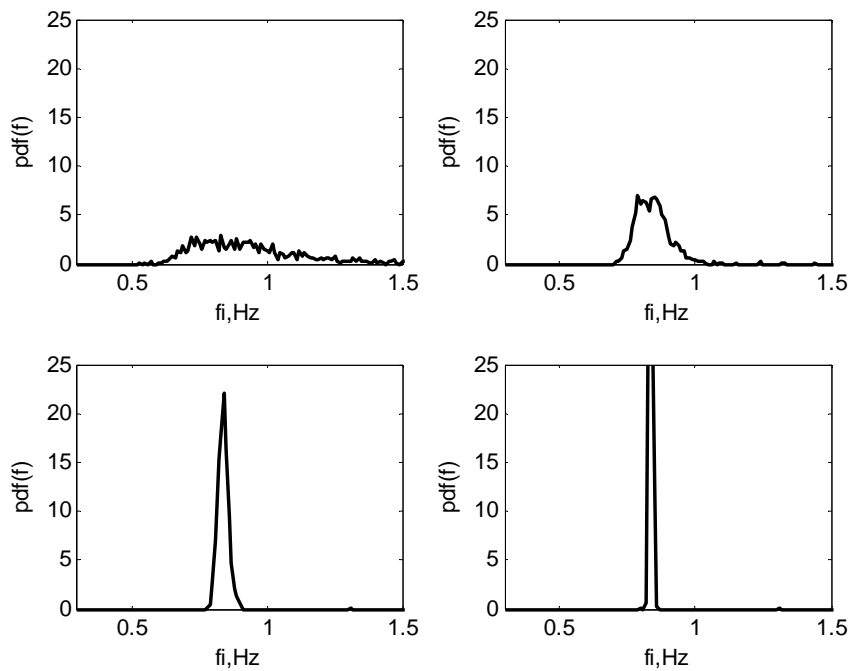


Fig. 3: Frequency tracking probability density function (PDF) for different SNRs in AWGN. From left to right, top to bottom: SNR=0, 10, 20, 30 dB.

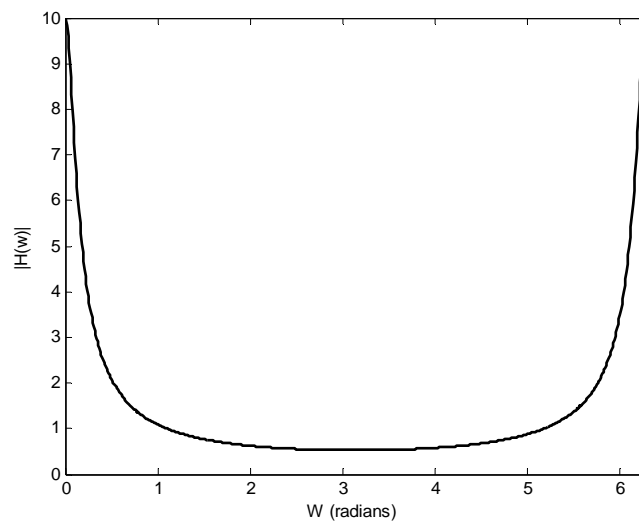


Fig. 4: 1st order IIR LPF magnitude transfer function $[H(z)=z/(z-0.9)]$. The digital frequency $w=\pi$ (or 3.14 rad) corresponds to 0.415 Hz and 2π (or 6.28 rad) corresponds to 0.83 Hz. Notice the peak is at 0 and 2π corresponding to 0.83 Hz.

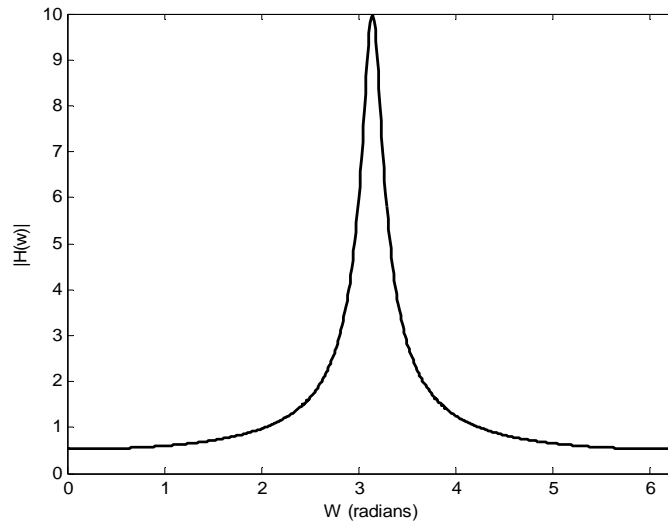


Fig. 5: 1st order IIR HPF magnitude transfer function $[H(z)=z/z+0.9]$. Notice the peak at π or 3.14 radians corresponding to 0.415 Hz.

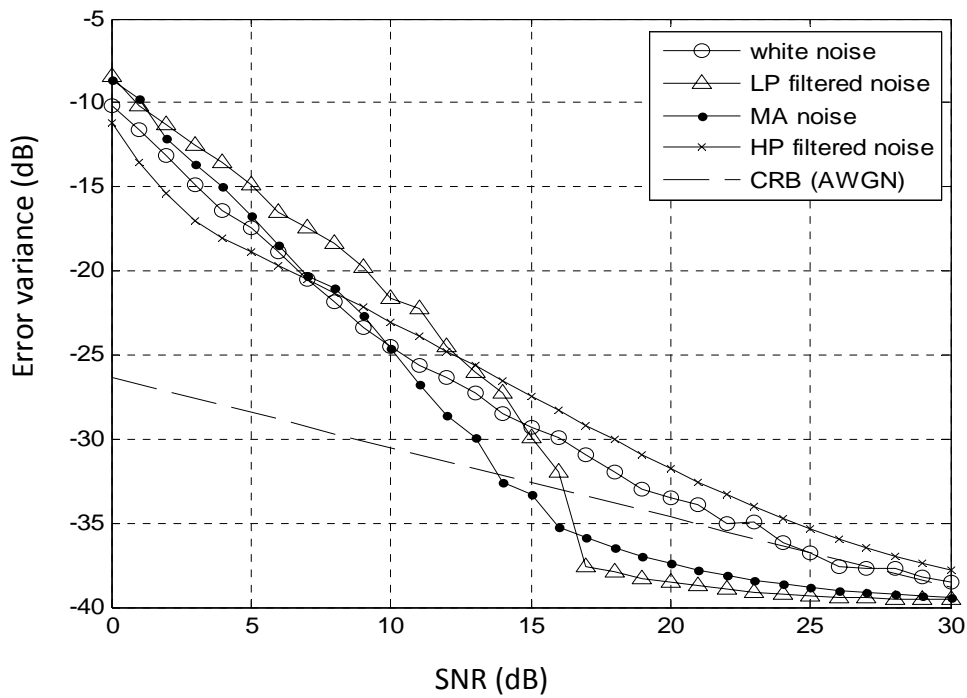


Fig. 6: Frequency error variance (dB) versus SNR (dB) for SPDLL estimator and for different noise types. The CRB lower bound is shown for the white noise case.

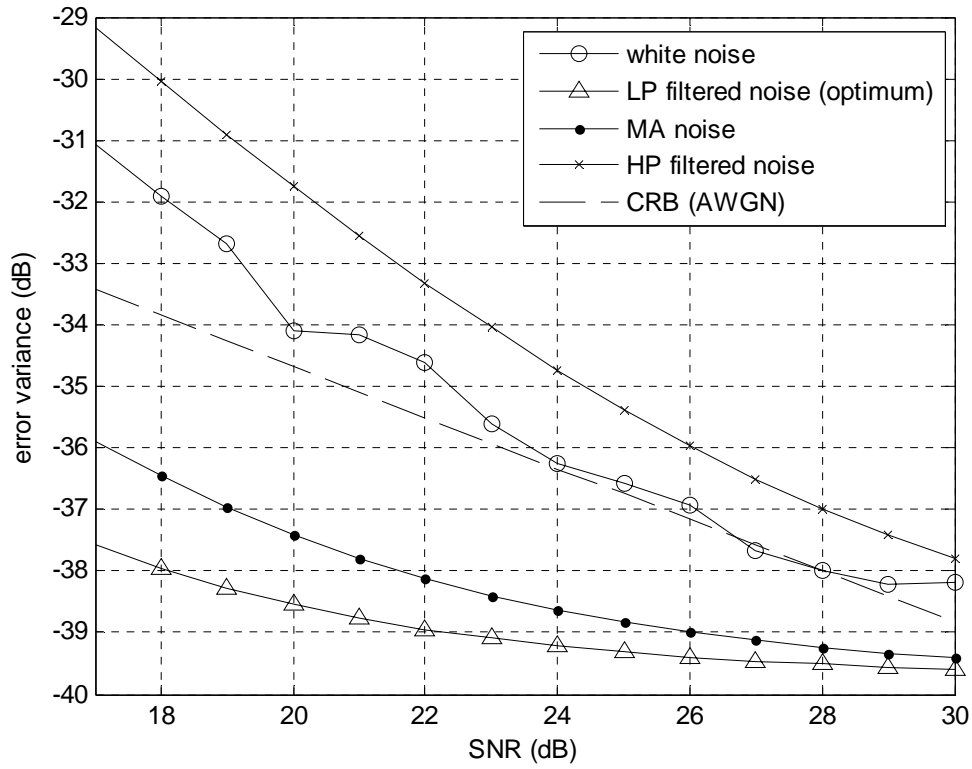


Fig. 7: Frequency error variance (dB) versus SNR (dB) for SDPLL estimator and for different noise types with emphasis on the moderate-to-high SNR region. The CRB lower bound is shown for the white noise case.