# Vibration Control Analysis of a Smart Flexible Cantilever Beam Using Smart Material

**Dr. Wedad Ibraheem Majeed**
University of Baghdad
Eng. Coll. Mech. Dep.

**Dr. Shibly Ahmed Al-Samarraie**
University of Technology
Mechatronic Dep.

**Mohanad Mufaq AL-SAIOR**
University of Baghdad
Eng. Coll. Mech. Dep.

## ABSTRACT

This paper features the modeling and design of a pole placement and output Feedback control technique for the Active Vibration Control (AVC) of a smart flexible cantilever beam for a Single Input Single Output (SISO) case. Measurements and actuation actions done by using patches of piezoelectric layer, it is bonded to the master structure as sensor/actuator at a certain position of the cantilever beam.

The smart structure is modeled based on the concept of piezoelectric theory, Bernoulli -Euler beam theory, using Finite Element Method (FEM) and the state space techniques. The number of modes is reduced using the controllability and observability grammians retaining the first three dominant vibratory modes, and for the reduced system, a control law is designed using pole placement and output feedback techniques.

The analyzed case studies concern the vibration reduction of a cantilever beam with a collocated symmetric piezoelectric sensor/actuator pair bonded on the surface. The transverse displacement time history, for an initial displacement field at the free end, is evaluated. Results are compared with other works, and the control design shows that Pole Placement method is an effective method for vibration suppression of the beam and settling time reduction.

**Keywords**: vibration, active vibration control, smart material, cantilever beam.

## الخلاصة:

هذا البحث يصف التمثيل الرياضي والتصميم لطريقة السيطرة الخاصة بتغيير موقع الجذور واستخدام الارجاع للمتغيرات الناتجة وذلك لاستخدام السيطرة الفاعلة على الاهتزازات لعتبة ذكية وباستخدام حالة ادخال واخراج مفردة. عمليات التوجيه و التحسس لازاحة العتبة تتم بواسطة موجهات و متحسسات مصنوعة من المواد الذكية على شكل ملصقات باحجام محددة تكون ملصوقة على سطح العتبة الرئيسية و بمواقع محددة.

تمت نمذجة العتبة الذكية باستخدام نظرية المواد الكهربية الذكية، نظرية برنولي – اويلر للعتبات و نظرية العناصر المحددة و حل المعادلات باستخدام نظرية الحالة المخمنة. عدد الاطوار الكلي للنموذج تمت تقليصها الى الثلاث اطوار الاساسية و المسيطرة باستخدام نظرية محددات معاملات السيطرة و التخمين. تم تصميم قانون للسيطرة للنموذج المقلص الجديد حيث استخدمت نظرية ابدال الجذور للنموذج الرياضي و السيطرة باستخدام ارجاع المخرجات.

درست الحالة في هذا البحث تقليص الاهتزاز لعتبة مصنوعة من الالمينوم باستخدام زوج واحد من المواد الذكية ملصوق بشكل متناظر على سطحي العتبة. الاهتزاز في العتبة منتج بواسطة ازاحة راس العتبة من الجهة السائبة بمسافة 1 سم. و تم تسجيل هذا الاهتزاز بالنسبة للزمن اضافة الى ذلك تمت مقارنة النتائج باعمال اخرى. اظهرت الدراسة ان نظرية السيطرة الفاعلة للاهتزاز فعالة لاخماد الاهتزازات في هكذا هياكل.

## INTRODUCTION

It is desired to design lighter mechanical systems carrying out higher workloads at higher speeds. However, the vibration may become prominent factor in this case. Smart materials and active control methods can be used to eliminate the undesired vibration. This combination of smart material and active vibration control paid considerable attention in the last decade especially in the space structures application.

Since the middle of the 19th century and due to the importance of providing a lighter, strong and vibration resisting structures specially in aero-structure and space application, the smart materials and active vibration control methods considered important for these application [Anna-Maria R. McGowan, 1998].

### 1.1. Piezoelectric Materials (Smart Materials)

In the year 1880, the brothers Pierre and Jacques Curie discovered the direct piezoelectric effect at tourmaline crystals. The inverse piezoelectric effect was predicted in 1881 by M. G. Lippmann based on thermodynamic considerations and afterwards confirmed experimentally by the brothers Curie [R.G. Ballas, 2007].

**T.C. Manjunath and B. Bandyopadhyay (2007)** [T.C. Manjunath, 2007] presented the modeling and design of a (FOS) Feedback control technique for the Active Vibration Control of a smart flexible aluminium Cantilever beam for a (SISO) case. The entire structure modeled in state space form using the concept of piezoelectric theory, Euler-Bernoulli beam theory and FEM techniques. The conclusions drawn for the best performance and for smallest magnitude of the control input required to control the vibrations of the beam. The MATLAB program used in results simulations.

**Tamara Nestorović and Miroslav Trajkov (2010)** [Tamara Nestorović, 2010] studied the piezoelectric applications in active vibration and noise attenuation in mechanical and civil engineering involving subsequent steps of modeling, control, simulation, experimental verification and implementation.

**Deepak Chhabra, Pankaj Chandna , Gian Bhushan (2011)** [Deepak Chhabra, 2011] designed the state/output feedback control by Pole placement technique and LQR optimal control approach achieve the desired control for the same smart cantilever beam in [T.C. Manjunath, 2007]. Sufficient vibrations attenuation was achieved. Pole placement technique is used to obtain the desired Eigen values of controlled system.

**Meysam Chegini, Milad Chegini and Hadi Mohammadi (2011)** [Meysam Chegini, 2011] used the PID control technique in their paper instead of (FOS) Feedback controller in (T.C. Manjunath and B. Bandyopadhyay (2007)) for the same smart cantilever beam, concluding that it is possible to implement classical controllers, such as PID and PI to control the amplitude and time of the vibration.

The analysis performed by **Gergely Takács (2011)** [Gergely Takács, 2011] presents the smart cantilever beam but with digital feedback control loops inside ANSYS transient simulations using the proprietary macro language of the software package. The close agreement of the closed-loop simulation results and laboratory measurements indicates the potential to use ANSYS for the preliminary prototyping of active vibration control systems (AVC).

### 1.2. Smart Structures

Piezoelectric materials could be divided, from structural viewpoint, into ceramic and polymeric forms. The most popular piezoelectric ceramics (or in short, piezoceramics) are compounds of lead zirconate titanate (PZT), the properties of which can be optimized to suit specific applications by appropriate adjustment of the zirconate–titanate ratio.

The polymeric form of the piezoelectric materials as polyvinylidene fluoride (PDVF) having low stiffness and electromechanical coupling coefficients (when compared to ceramics like PZT, for instance) [Nader Jalili, 2010].

Structures with added functionality over and above the conventional purpose of providing strength by reinforcement or stiffness may be regarded as smart. Smart or adaptive structures, based on using a small change in the structure geometry at critical locations induced by internally generated control signals, can result in a non-linear amplification of the shape, stiffness or strength, and so the structure will adapt to a functional need. In practice, smart structures may be classified depending on their functionality and adaptation to the changing situation:

1. *Passive smart*
2. *Active smart*
3. *Intelligent.*

## 2. MODELING OF SMART CANTILEVER BEAM

The smart structure is modeled based on the concept of piezoelectric theory, Bernoulli - Euler beam theory, using Finite Element Method (FEM).

### 2.1. Displacement Functions

A beam element is considered with two nodes at its end. Each node is having two degree of freedom (DOF). The shape functions of the element are derived by considering an approximate solution and by applying boundary conditions. The mass and stiffness matrix is derived using shape functions for the beam element. Mass and stiffness matrix of piezoelectric (sensor/actuator) element are similar to the beam element. To obtain the mass and stiffness matrix of smart beam element that consists of two piezoelectric materials and a beam element, all the three matrices added.

FEM assembly of beam element and smart beam element models the cantilever beam. The last two row's two elements of first matrix are added with first two row's two element of next matrix. The global mass and stiffness matrix is formed. The boundary conditions are applied on the global matrices for the cantilever beam. The first two rows and two columns should be deleted as one end of the beam is fixed. The actual response of the system, i.e., the tip displacement u (x,t) is obtained for all the various models of the cantilever beam with and without the controllers[Deepak Chhabra, 2011].

A beam element of length $L_b$ with two DOFs at each node i.e. translation and rotation is considered[T.C. Manjunath, 2007].

$$W(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3, \quad (1)$$

Application of the boundary condition of the cantilever beam yields:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \frac{1}{l_b^3} \begin{bmatrix} l_b^3 & 0 & 0 & 0 \\ 0 & l_b^3 & 0 & 0 \\ -3l_b & -2l_b^2 & 3l_b & -l_b^2 \\ 2 & l_b^3 & -2 & l_b^3 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}, \quad (2)$$

Substituting the constants obtained from (2) into (1) and by rearranging the terms, the final form for W(x) is obtained as:

$$w(x) = [n]^T [q], \quad (3)$$

where $[n]$ gives the shape function as, [Ranjan Vepa, 2010]:

$$[n] = \begin{bmatrix} 1 - 3\frac{(x-x_i)^2}{l_b^2} + 2\frac{(x-x_i)^3}{l_b^3} \\ (x-x_i) - 2\frac{(x-x_i)^2}{l_b} + 2\frac{(x-x_i)^3}{l_b^2} \\ 3\frac{(x-x_i)^2}{l_b^2} - 2\frac{(x-x_i)^3}{l_b^3} \\ -\frac{(x-x_i)^2}{l_b} + 2\frac{(x-x_i)^3}{l_b^2} \end{bmatrix}, \quad (4)$$

and $[q]$ is the vector of displacements and slopes (nodal displacement vector) and is given by:

$$[q] = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}, \quad (5)$$

## 2.2. DYNAMIC EQUATION OF THE BEAM ELEMENT

The equation of motion of the regular beam element is obtained by the lagrangian equation:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] + \left[\frac{\partial U}{\partial q_i}\right] = [F_i], \quad (6)$$

As:

$$M^b \ddot{q} + K^b q = f^b(t), \quad (7)$$

The strain energy $U$ and the kinetic energy $T$ for the beam element with uniform cross section in bending is obtained as:

$$U = \frac{E_b I_b}{2} \int_{l_b} \left[\frac{\partial^2 w}{\partial x^2}\right] dx, = \frac{E_b I_b}{2} \int_{l_b} [w''(x,t)]^T [w''(x,t)] dx, \quad (8)$$

$$T = \frac{\rho_b A_b}{2} \int_{l_b} \left[\frac{\partial w}{\partial t}\right]^2 dx, = \frac{\rho_b A_b}{2} \int_{l_b} [\dot{w}(x,t)]^T [\dot{w}(x,t)] dx, \quad (9)$$

Element in its explicit form is obtained as:

$$\frac{\rho_b A_b}{420} \begin{bmatrix} 156 & 22l_b & 54 & -13l_b \\ 22l_b & 4l_b^2 & 13l_b & -3l_b^2 \\ 54 & 13l_b & 156 & -22l_b \\ -13l_b & -3l_b^2 & -22l_b & 4l_b^2 \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \frac{E_b I_b}{l_b} \begin{bmatrix} \frac{12}{l_b^2} & \frac{6}{l_b} & -\frac{12}{l_b^2} & \frac{6}{l_b} \\ \frac{6}{l_b} & 4 & -\frac{6}{l_b} & 4 \\ -\frac{12}{l_b^2} & -\frac{6}{l_b} & \frac{12}{l_b^2} & -\frac{6}{l_b} \\ \frac{6}{l_b} & 4 & -\frac{6}{l_b} & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}, \quad (10)$$

where $\mathbf{F_1}$ and $\mathbf{F_2}$ are the forces at the nodes 1 and 2, $\mathbf{M_1}$ and $\mathbf{M_2}$ are the bending moments acting at the nodes 1 and 2 respectively.

The piezoelectric beam element shown in Figure (1) is obtained by sandwiching the regular beam element between two piezoelectric thin layers of thickness $t_a$ or $t_s$.

The bottom layer acts as a sensor and the upper layer acts as an actuator. Similar to the equation (10) obtained for a regular beam element, the Lagrangian equation of motion of the piezoelectric beam element is obtained as,

$$M^p\ddot{q} + K^p q = f^p(t), \qquad (11)$$

where $M^p$ and $K^p$ are the mass and stiffness matrices of the piezoelectric element and is obtained as, [Daniel J. Inman, 2006]:

$$[M^p] = \frac{\rho A}{420}\begin{bmatrix} 156 & 22l_b & 54 & -13l_b \\ 22l_b & 4l_b^2 & 13l_b & -3l_b^2 \\ 54 & 13l_b & 156 & -22l_b \\ -13l_b & -3l_b^2 & -22l_b & 4l_b^2 \end{bmatrix}, (12)$$

and

$$[K^p] = \frac{EI}{l_b}\begin{bmatrix} \frac{12}{l_b^2} & \frac{6}{l_b} & -\frac{12}{l_b^2} & \frac{6}{l_b} \\ \frac{6}{l_b} & 4 & -\frac{6}{l_b} & 4 \\ -\frac{12}{l_b^2} & -\frac{6}{l_b} & \frac{12}{l_b^2} & -\frac{6}{l_b} \\ \frac{6}{l_b} & 4 & -\frac{6}{l_b} & 4 \end{bmatrix}, \qquad (13)$$

where :

$$EI = E_b I_b + 2E_p I_p, \qquad (14)$$

$$I_p = \frac{1}{12}bt_a^3 + bt_a\left(\frac{t_a+t_b}{2}\right)^2, \qquad (15)$$

$$\rho A = b(\rho_b t_b + 2\rho_p t_p), \qquad (16)$$

## 2.3. Piezoelectric strain rate sensors and actuators

The sensor equation is derived from the direct piezoelectric equation. The electric displacement developed on the sensor surface is directly proportional to the stress acting on the sensor. If the poling is done along the thickness direction of the sensors with the electrodes on the upper and the lower surfaces, the electric displacement D is given by, [Michael R. Hatch, 2001]:

$$D_z = d_{31} E_p \varepsilon_x = e_{31}\varepsilon_x, \qquad (17)$$

where $d_{31}$ is the piezoelectric constant, $e_{31}$ is the piezoelectric stress / charge constant, $E_p$ is the young's modulus and $\varepsilon_x$ is the strain that is produced.

### 2.3.1. Sensor equation

The total charge $Q(t)$ developed on the sensor surface is the spatial summation of all the point charges developed on the sensor layer. Thus, the expression for the current generated is obtained as, [Ranjan Vepa, 2010]:

$$i(t) = \frac{d\,Q(t)}{dt} = \frac{d}{dt}\int_A e_{31}\,\varepsilon_x\,dA = ze_{31}b\int_{x_i}^{x_i+l_p} n_1{}^T\dot{q}\,dx, \qquad (18)$$

where, $z = \frac{t_b}{2} + t_a$, and $n_1$ is the second spatial derivative of the shape function given in (Appendix 1) [T.C. Manjunath, 2007].

After converting the current $i$ using signal conditioning device with gain $G_c$ into voltage $V_s$, thus the sensor output voltage is obtained as,:

$$V_s(t) = G_c\,e_{31}\,z\,b\int_{x_i}^{x_i+l_p} n_1^T\,\dot{q}\,dx, \qquad (19)$$

Substituting for $n_1$ and $\dot{q}$ in equation (19) and simplifying, we get the sensor voltage for a 2-nodes finite element as:

$$V_s(t) = [G_c e_{31} zb \;\; 0 \;\; -G_c e_{31} zb \;\; 0]\begin{bmatrix}\dot{w}_1\\ \dot{\theta}_1\\ \dot{w}_2\\ \dot{\theta}_2\end{bmatrix}, (20)$$

$$V_s(t) = P^T\dot{q}, \qquad (21)$$

where $\dot{q}$ is the time derivative of the modal coordinate vector $q$ , $P$ is a constant vector which depends on the type of sensor, its characteristics and its location on the beam.

### 2.3.2. Actuator equation

The actuator strain is derived from the converse piezoelectric equation. The strain developed a $\varepsilon_a$ on the actuator layer is given by

$$\varepsilon_a = d_{31} E_f, \qquad (22)$$

where $d_{31}$ and $E_f$ are the piezoelectric strain constant and the electric field respectively. When the input to the piezoelectric actuator $V_a(t)$, is applied in the thickness direction $t_a$, the electric field, $E_f$ which is the voltage applied $V_a(t)$ divided by the thickness of the actuator $t_a$ ; and the stress, $\sigma_a$ which is the actuator strain multiplied by the young's modulus $E_p$ of the piezoelectric actuator layer are given by

$$E_f = \frac{V_a(t)}{t_a}, \qquad (23)$$

$$\sigma_a = E_p\, d_{31} \frac{V_a(t)}{t_a}, \qquad (24)$$

The strain developed on the actuator layer is directly proportional to the electric field $E_f$, The resultant moment $M_a$ acting on the beam is thus determined by integrating the stress through the structural thickness as, [T.C. Manjunath, 2007]:

$$M_a = E_p\, d_{31}\, \bar{z}\, V_a(t), \qquad (25)$$

where $\bar{z} = \frac{t_a + t_b}{2}$ is the distance between the neutral axis of the beam and the piezoelectric layer [T.C. Manjunath, 2007]. Finally, the control force applied by the actuator is obtained as:

$$f_{ctrl} = E_p\, d_{31} b\, \bar{z} \int_{x_i}^{x_i + l_p} n_2\, dx\, V_a(t), \qquad (26)$$

$$V_s(t) = \begin{bmatrix} 0 & 0 & 0 & -E_p\, d_{31} b\, \bar{z} \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ \dot{\theta}_1 \\ \dot{w}_2 \\ \dot{\theta}_2 \end{bmatrix}, \qquad (27)$$

$$f_{ctrl} = h\, V_a(t) = h\, u(t), \qquad (28)$$

### 2.4. Formulation

The dynamic equation of the smart structure is obtained by using both the regular and piezoelectric beam elements given by equations (10, 12 and 13), the mass and stiffness of the bonding or the adhesive between the master structure and the sensor / actuator pair is neglected.

After assembling the general mass and stiffness matrices and including the generalized structural damping matrix $[C] = \alpha * [M] + \beta * [K]$, where $\alpha$ and $\beta$ are the frictional damping constant and the structural damping constant, and applying the cantilever beam boundary condition, the system equation of motion for the 4-element cantilever beam is:

$$[M]_{8x8}\, \ddot{q}(t) + [C]_{8x8}\, \dot{q}(t) + [K]_{8x8}\, q(t) = f, \qquad (29)$$

and

$$f = f_{ext} + f_{ctrl}, \qquad (30)$$

for free vibration condition $f_{ext}$ equal to zero, so the remaining applied force on the system is the controlling force exerted by the controller.

### 2.5. State Space model of the smart cantilever beam

The equation (29) could be written in state space form as follows, Let the states of the system be defined as,

$$q = x, \qquad (31)$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x, \qquad (32)$$

$$\dot{q} = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \qquad (33)$$

$$\ddot{q} = \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \qquad (34)$$

Substituting equations (31 – 34) in (29), we get,:

$$[M]\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + [C]\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + [K]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f_{ctrl}, \quad (35)$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = -[M]^{-1}[C]\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$
$$-[M]^{-1}[K]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [M]^{-1}f_{ctrl}, \qquad (36)$$

and finally written in state equation form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}h \end{bmatrix}u(t), \qquad (37)$$

$$\dot{x} = A\,x(t) + B\,u(t), \qquad (38)$$

The output equation (sensor equation) for a SISO case is given by:

$$y(t) = V_s(t) = P^T\dot{q} = P^T\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \qquad (39)$$

$$y(t) = \begin{bmatrix} 0 & P^T \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad (40)$$

$$y(t) = C\,x(t) + D\,u(t), \qquad (41)$$

The SISO state space model (state equation and output equation (41)) of the smart cantilever beam finally is given by equations (38) and (41), with,

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}_{16\times16}, (42)$$

$$B = \begin{bmatrix} 0 \\ M^{-1}h \end{bmatrix}_{16\times1}, \qquad (43)$$

$$C = \begin{bmatrix} 0 & P^T \end{bmatrix}_{1\times16}, \qquad (44)$$

$$D = 0, \qquad (45)$$

## 3. ACTIVE VIBRATION CONTROL

Active vibration control (AVC) is an important problem in structures. One of the ways to tackle this problem is to make the structure smart, intelligent, adaptive and self-controlling by making use of the smart material.

### 3.1 Output Feedback Control

The vibrations of many structures and devices are controlled by sophisticated control methods. Examples of the use of feedback control to remove vibrations range from machine tools to tall buildings and large spacecraft. This method considered, as one of most popular way to control the vibrations of a structure by measuring the position and velocity vectors of the structure and to use that information to drive the system in direct proportion to its positions and velocities [Daniel J. Inman, 2006].

Consider the following dynamical system form,

$$M\ddot{\boldsymbol{q}}(t) + (C + \mathrm{G})\dot{\boldsymbol{q}}(t) + K\boldsymbol{q}(t) = \mathrm{f}(t), \quad (46)$$

Where, $\boldsymbol{q}$ represent a generalized coordinate that may not be an actual physical coordinate or position but is related, $M = M^T$is the mass, or inertia, matrix, $C = C^T$is the viscous damping matrix, $G = G^T$is gyroscopic matrix, $K = K^T$is stiffness matrix, and The $n$ vector $f= f(t)$ represents applied external forces and is also time varying.

A condition to ensure stability, M, C and K is positive definite, such that $(x^T A\, x < 0)$ for all nonzero real vectors $x$, and $(x^T A\, x = 0)$ if and only if $x$ is zero.

When Output Feedback Control done for the above system, Equation (46) becomes

$$M\ddot{q}(t) + (C + G)\dot{\boldsymbol{q}}(t) + K\boldsymbol{q}(t) = -K_p\boldsymbol{q}(t) - K_v\dot{\boldsymbol{q}}(t) - \mathrm{f}(t), \quad (47)$$

Here, $K_p$ and $K_v$ are called feedback gain matrices [Daniel J. Inman, 2006].

### 3.2 Model Order Reduction

It is necessary to reduce the order of a model before performing a control analysis and designing.

The approach taken for reduction the order of a given model based on deleting the coordinates, or modes, that are the least controllable and observable. The idea here is that controllability and observability of a state (coordinate) are indications of the contribution of that state (coordinate) to the response of the structure, as well as the ability of that coordinate to be excited by an external disturbance[Daniel J. Inman, 2006].

A useful measure is provided for asymptotically stable system of the form given by Equations (38) and (41) by defining the controllability grammian, denoted by $W_C$, as

$$W_C^2 = \int_0^\infty e^{At}\, BB^T e^{A^T t} dt, \quad (48)$$

and the observability grammian, denoted by $W_O$, as:

$$W_O^2 = \int_0^\infty e^{A^T t}\, C^T C e^{At} dt, \quad (49)$$

If the system is controllable (or observable), the matrix $W_C$ (or $W_O$) is nonsingular. These grammians characterize the degree of controllability and observability by quantifying just how far away from being singular the matrices $W_C$ and $W_O$ are. This is equivalent to quantifying rank deficiency. The most reliable way to quantify the rank of a matrix is to examine the singular values of the matrix. [Daniel J. Inman, 2006]

Let the matrix $\mathbf{P}$ denote a linear similarity transformation, which when applied to Equations (38) and (41) yields the equivalent system

$$\dot{x}' = A'\, x' + B'\, u, \quad (50)$$

$$y = C'\, x', \quad (51)$$

These two equivalent systems are related by

$$x' = \mathbf{P}\, x, \quad (52)$$

$$A' = \mathbf{P}^{-1} A\ \mathbf{P}, \quad (53)$$

$$B' = \mathbf{P}^{-1} B, \quad (54)$$

$$C' = C\,\mathbf{P}, \qquad (55)$$

Here, matrix $\mathbf{P}$ can be chosen such that the new grammians defined by

$$W_C' = \mathbf{P}^{-1} W_C\ \mathbf{P}, \qquad (56)$$

and,

$$W_O' = \mathbf{P}^{-1} W_O\ \mathbf{P}, \qquad (57)$$

are equal and diagonal. That is,

$$W_C' = W_O' = \Lambda_W = \mathrm{diag}[\sigma_1\ \sigma_2 \cdots \sigma_{2n}]$$

where the numbers $\sigma_i$ are the singular values of the grammians and are ordered such that

$$\sigma_i > \sigma_{i+1}, \qquad i = 1,2,\dots,2n-1$$

The MATLAB program offer a good algorathems for the system order reduction using the controllability and observability gramians called Control System Toolbox. The command (balreal) used to perform the balanced relization based on controllability and observability gramians, where the command (modred) used for system order reduction [Michael R. Hatch, 2001].

### 3.3 Controller Design using Pole Placement and output feedback techniques

Stability of a system are closely related to the location of poles or eigenvalues of the system. Pole placement can be achieved by feedback control. The poles of this system are eigenvalues of $A$.

Using the state feedback

$$u = -\mathbf{K}_{ctrl}\, x, \qquad (58)$$

Where $(\mathbf{K}_{ctrl}\, x)$ is linear state feedback.

Under this feedback control, the controlled closed loop system is given by[Deepak Chhabra, 2011]:

$$\dot{x} = A_{ctrl}\, x(t) + Er(t), \qquad (59)$$

Where:

$$A_{ctrl} = (A - B\mathbf{K}_{ctrl}), \qquad (60)$$

To find out the value of feedback gain matrix $\mathbf{K}_{ctrl}$, A characteristic equation of the system is considered as:

$$\varphi(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots$$
$$+ a_1 s + a_0, \qquad (61)$$

The poles of the controlled system is in the desired locations represented by the desired characteristic equation as:

$$\varphi(s) = s^n + a_{n-1}'s^{n-1} + a_{n-2}'s^{n-2} + \cdots$$
$$+ a_1' s + a_0', \qquad (62)$$

This can be achieved by letting the feedback matrix be:

$$\mathbf{K}_{ctrl} = [k_0\quad k_1\quad \dots\quad k_{n-1}]$$
$$= [a_0 - a_0'\quad a_1 - a_1'\quad \dots\quad a_{n-1} - a_{n-1}'], (63)$$

The MATLAB command (place) in Control System Toolbox could be used to

computes the feedback gain matrix $\mathbf{K}_{ctrl}$ that achieves the desired closed-loop pole locations. The algorathem used in this command uses the extra degrees of freedom to find a solution that minimizes the sensitivity of the closed-loop poles to perturbations in $A$ or $B$ matrices, and it optimizes the choice of eigenvectors for a robust solution.

## 4. NUMERICAL SIMULATION AND RESULTS

Figure (2) illusterate the smart canteliver beam with the embaded sensor and actuator layers, showing the devisions considered in the FEM solution.

The dimensions of the piezoelectric patch are given in Table (1) and Table (2).

The bode plot of the original system and the balanced reduced system is shown if figure(3)

The resulted natural frequencies (N.F) from the numerical simulation is matching the reference N.F, also A complete match was found between the original system model and the reduced system model for the first three modes.

The tip response of smart cantilever beam for free vibration case with initial tip displacement of (1 cm) is shown in figure (4) for the case at which the sensor/actuator patches placed onto position of element 2. In addition, the free vibration tip response for the case at

which the patches placed onto element 4 is shown in figure(5).

After applying the control law the new closed loop system bode plot is shown in figure(6) for the case of the patches onto element 2, also the effect of the control action on the cantilever response for the case of the patches onto element 2 is represented in figure(7). Moreover, for the case of the patches onto element 4 is represented in figure (8).

The patches position affects the free response of the beam as shown in figures (4) and (5). In addition, the case in which the piezoelectric patches placed onto the position of element (2) is better than the position of element (4) in vibration suppression using the proposed control law.

## 5. Conclusions

In conclusion, it could be said that the vibration suppressed successfully for the cantilever beam using 2 piezoelectric patches of piezoelectric material as sensor and actuator.

It is found out that, the Pole Placement method is an applicable way for the active vibration control vibration

The reduction method used was very effective.

On the other hand, it is possible to implement optimal control techniques to control the amplitude and time of the vibration.

## REFERENCES


- Anna-Maria R. McGowan, et. al., NASA Langley Sesearch Center, Aeroelasticity and Structural Dynamics branches, “Aeroservoelatic and structural dynamics research on smart structruras conducted at NASA Langley Research Center”, Hampton, Virginia, May, 1998.
- B. Bandyopadhyay, T.C. Manjunath, M. Umapathy “Modeling, Control and Implementation of Smart Structures, A FEM - State Space Approach”, Springer

Berlin Heidelberg, New York, USA, 2007.

- Daniel J. Inman, "Vibration with Control", John Wiley & Sons Ltd, England, 2006.
- Deepak Chhabra, Pankaj Chandna and Gian Bhushan," Design and Analysis of Smart Structures for Active Vibration Control using Piezo-Crystals", International Journal of Engineering and Technology Volume-1 No.3, December, 2011.
- Gergely Takács, "Experimental Verification Of The Closed-Loop Response Of An Active Vibration Control System Modeled In ANSYS", The 4th International conference - Modeling Of Mechanical And Mechatronic Systems, Faculty of Mechanical engineering, Technical university of Košice, Slovak, 2011.
- Meysam Chegini, Milad Chegini and Hadi Mohammadi," Vibration Suppression of a Beam of Structures", IACSIT International Journal of Engineering and Technology, Vol.3, No.2, 2011.
- Michael R. Hatch, "Vibration simulation using MATLAB and ANSYS", Chapman & Hall/CRC, 2001.
- Nader Jalili, "Piezoelectric-Based Vibration Control, From Macro to Micro/Nano Scale Systems", Springer Berlin Heidelberg, New York, USA, 2010.
- R.G. Ballas, "Piezoelectric Multilayer Beam Bending Actuators: Static and Dynamic Behavior and Aspects of Sensor Integration", Springer-Verlag Berlin Heidelberg, 2007.
- Ranjan Vepa, "Dynamics of Smart Structures", John Wiley and Sons Ltd. Publication, UK, 2010.
- Scott M. Newman, Thesis Titled "Active Damping Control of a Flexible Space Structure Using Piezoelectric Sensors and Actuators", NAVAL Postgraduate School, Monterey, California, 1992.
- T.C. Manjunath and B. Bandyopadhyay, "Control of Vibrations in Flexible Smart Structures using Fast Output Sampling Feedback Technique", International Journal of Information and Mathematical Sciences 3:2, 2007.
- Tamara Nestorović, Miroslav Trajkov, "Active Control of Smart Structures an Overall Approach", FACTA Universitatis, Architecture and Civil Engineering, Vol. 8, No 1, pp. 35 – 44, 2010.

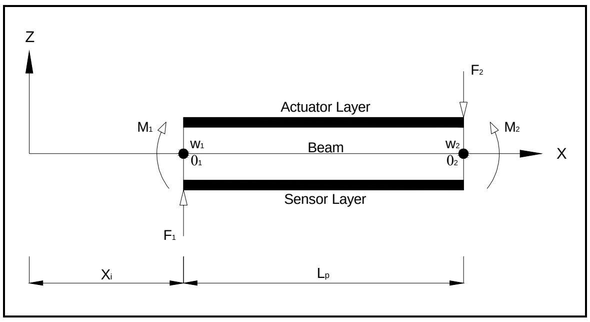


**Figure 1** Smart beam element

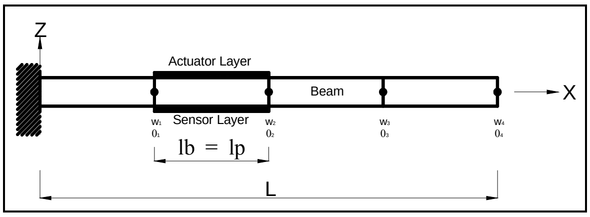


**Figure2** smart cantilever beam

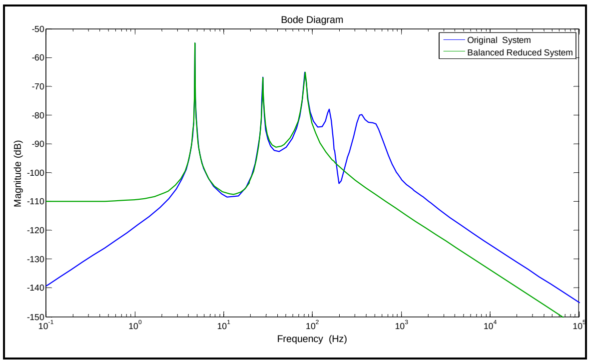


**Figure 2** Bode plot of the original system and the balanced reduced system

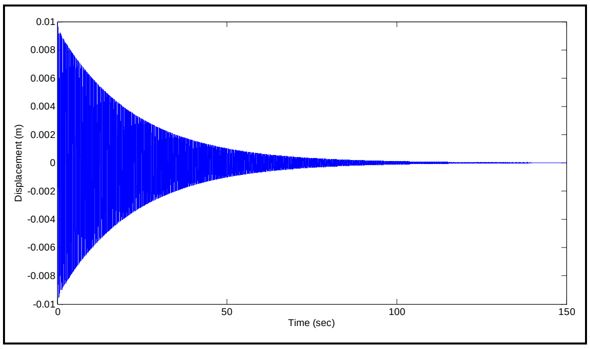


**Figure 3** Free Vibration tip response for sensor/actuator patches on to element 2

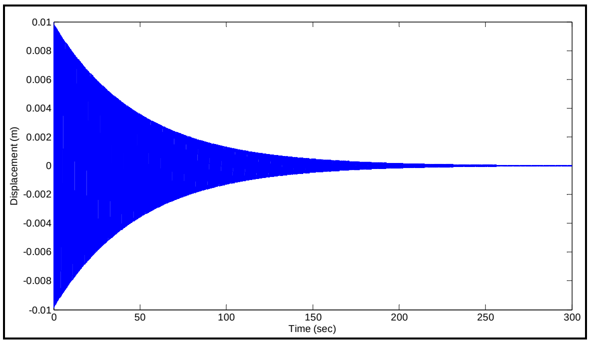


**Figure 4** Free Vibration tip response for sensor/actuator patches on to element 4

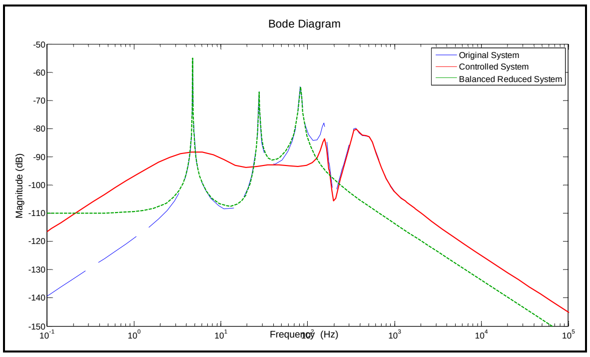


**Figure 5** Bode plot of the original system, the balanced reduced system and the new closed loop controlled system

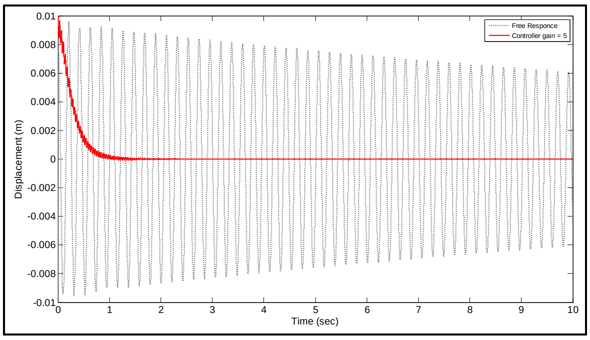


**Figure 6** AVC for beam with patch on element(2)

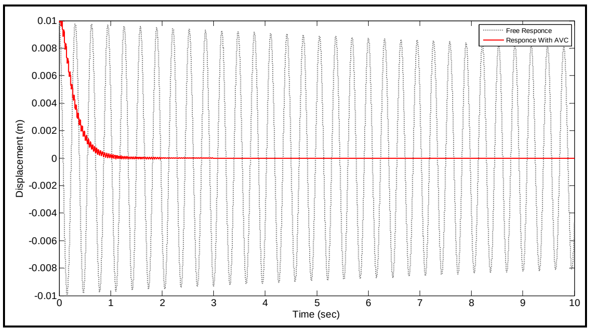


**Figure 7** AVC for beam with patch on element(4)

**Table (1)** properties of the flexible cantilever beam when the beam is divided into 4 elements [B.Bandyopadhyay, 2007]

| Parameter | Symbol | Numerical Value |
|---|---|---|
| Length (m) | $l_b$ | 0.03 |
| Width (m) | $b$ | 0.03 |
| Thickness (mm) | $t_b$ | 0.5 |
| Young's Modulus (Gpa) | $E_b$ | 193.06 |
| Density (kg/m3) | $\rho_b$ | 8030 |
| Damping constant used in (C) | $\alpha$ , $\beta$ | 0.001, 0.0001 |

**T able (2)** properties of the piezoelectric element when the beam is divided into 4 elements [B. Bandyopadhyay, 2007].

| Parameter | Symbol | Numerical Value |
|---|---|---|
| Length (m) | $l_p$ | 0.075 |
| Width (m) | $b$ | 0.03 |
| Thickness (mm) | $t_p$ | 0.35 |
| Young's Modulus (Gpa) | $E_p$ | 68 |
| Density (kg/m3) | $\rho_p$ | 7700 |
| Piezoelectric Stain constant (m/V) | $d_{31}$ | $125\times10^{-12}$ |
| Piezoelectric Stress constant (V.m/N) | $e_{31}$ | $10.5\times10^{-3}$ |

**Table (3)** shows the natural frequency in (Hz) and verification with the reference paper

| N.F. | Numerical result by MATLAB | N.F. Ref. [B. Bandyopadhyay, 2007] | Error (%) |
|---|---|---|---|
| $\omega_1$ | 4.738 | 4.73 | 0.00 |
| $\omega_2$ | 27.429 | 27.43 | 0.00 |
| $\omega_3$ | 81.868 | -- | -- |

**Table(4)** Symbols Description

| Syb. | **Description** | **Unit** |
|---|---|---|
| $\alpha$ | Fractional damping constant | |
| $\beta$ | Structural damping constant | |
| $\epsilon$ | Strain | |
| $\epsilon_x$ | Mechanical normal strain | |
| $v$ | Poisson's ratio | |
| $\rho_b$ | Density of the beam | $Kg/m^3$ |
| $\rho_p$ | Density of the piezoelectric patch | $Kg/m^3$ |
| $\rho A$ | Mass / unit length | |
| $\sigma_a$ | Stress in the actuator | |
| $\sigma$ | Stress | $N/m^2$ |
| $\omega$ | Natural frequency | Hz |
| $\xi$ | Damping ratio | |

## Appendix - 1 -

Spatial Derivative of The Shape Function of Bernoulli-Euler Beam,

$$[n]=\begin{bmatrix} 1-3\dfrac{(x-x_i)^2}{l_b^2}+2\dfrac{(x-x_i)^3}{l_b^3} \\ (x-x_i)-2\dfrac{(x-x_i)^2}{l_b}+2\dfrac{(x-x_i)^3}{l_b^2} \\ 3\dfrac{(x-x_i)^2}{l_b^2}-2\dfrac{(x-x_i)^3}{l_b^3} \\ -\dfrac{(x-x_i)^2}{l_b}+2\dfrac{(x-x_i)^3}{l_b^2} \end{bmatrix}=[n_3]$$

$$[n_1]=[n_2']=\begin{bmatrix} -\dfrac{6}{l_b^2}+12\dfrac{(x-x_i)}{l_b^3} \\ -\dfrac{4}{l_b}+12\dfrac{(x-x_i)}{l_b^2} \\ \dfrac{6}{l_b^2}-12\dfrac{(x-x_i)}{l_b^3} \\ -\dfrac{2}{l_b}+6\dfrac{(x-x_i)}{l_b^2} \end{bmatrix}$$

$$[n_2]=[n_3']=\begin{bmatrix} -6\dfrac{(x-x_i)}{l_b^2}+6\dfrac{(x-x_i)^2}{l_b^3} \\ 1-4\dfrac{(x-x_i)}{l_b}+6\dfrac{(x-x_i)^2}{l_b^2} \\ 6\dfrac{(x-x_i)}{l_b^2}-6\dfrac{(x-x_i)^2}{l_b^3} \\ -2\dfrac{(x-x_i)}{l_b}+3\dfrac{(x-x_i)^2}{l_b^2} \end{bmatrix}$$