



Vibration Measurement and Analysis of knee-Ankle-Foot Orthosis (KAFO) Metal-Metal type

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ABSTRACT

This paper deals with calculate stresses in Knee-Ankle-Foot-Orthosis as a result of the effect vibration during gait cycle for patient wearing KAFO .Experimental part included measurement interface pressure between KAFO and leg due to action muscles and body weigh on Orthosis. also measurement acceleration result from motion of defected leg by accelerometer .Results of Experimental part used input in theoretical part so as to calculate stresses result from applying pressure and acceleration on KAFO by engineering analysis program ANSYS 14.Results show stresses values in upper KAFO greater than lower KAFO that is back to muscles more effective in thigh part lead to recoding pressure higher than pressure in shank part.

Keywords: knee, ankle, foot, orthosis, pressure, stress, acceleration

قياس وتحليل الاهتزازات لمسند نوع قدم كاحل ركبة

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الخلاصة:

هذا البحث يتناول حساب الاجهادات المتولدة في مسند نوع قدم كاحل ركبة نتيجة تأثير الاهتزازات الحاصلة أثناء دورة المشي للمريض .يتضمن الجزء العملي قياس ضغط التداخل بين المسند والساق بسبب تأثير العضلات ووزن الجسم على المسند .كذلك قياس التعجيل الناتج من حركة الساق المصابة بواسطة جهاز قياس التعجيل (accelerometer) .تستعمل نتائج الجزء العملي كمدخلات في الجزء النظري ليتم حساب الاجهادات الناتجة من تسليط الضغط والتعجيل على المسند ببرنامج التحليل الهندسي ANSYS 14.اظهرت النتائج ان قيمة الاجهادات الناتجة في اعلى المسند عند عضلات الفخذ اعلى من الاجهادات الناتجة في منطقة الساق والسبب يعود الى قيمة الضغط المسجلة في منطقة الفخذ اعلى من الضغط في منطقة الساق .

الكلمات الرئيسية: قدم ، كاحل ، ركبة ، مسند، تعجيل ،ضغط ، اجهاد.

1. INTRODUCTION

An orthosis is a device that is applied to a part of the body to correct deformity, improve function or relieve symptoms of a disease. The word is derived from ortho, meaning straight [Alaa'SaeedHadi].

Orthoses added to the body to stabilize or immobilize a body part, prevent deformity, protect against injury or to assist with function. They can be divided into different types based on their intended function [CIGNA HealthCare].

The term KAFO is an acronym that stands for Knee-Ankle-Foot Orthoses and describes the part of the body that the device encompasses. The device extends from the thigh to the foot and is generally used to control instabilities in the lower limb by maintaining alignment and controlling motion. Instabilities can be either due to skeletal problems: broken bones, arthritic joints, bowleg, knock-knee, knee hyperextension or muscular weakness and paralysis [Jumaa S. Chiad].

Typically, KAFOs are extremely mechanically simple and often have few moving parts. This simplicity is accompanied by ease of donning and durability but leaves functional abilities only partially improved. Historically, KAFOs have locked the knee joint, providing stance phase stability while preventing knee motion during swing. Alternatively, KAFOs with an eccentric knee joint allow knee motion during swing but provide limited stability during stance. Either design results in inefficient gait [Jumaa S. Chiad].

These devices use a knee joint that is mechanically stable during the stance phase but releases at swing phase. The resulting gait is much smoother than the gait with a traditional KAFO where the knee remains locked throughout the entire gait cycle. Continued engineering development and creativity will be required for evolution of these designs into viable components for use by patients with knee instability during stance [Jumaa S. Chiad].

The human being in the environments of modern technology has to endure stresses of many and

varied kinds of vibrations. These vibrations have significant effects on the physical and mental health. It has been observed that the stresses imposed by vibration have produced changes in the normal functions of the human body. The Whole body vibration can cause severe motion sickness which depends on the mechanical properties of the human body [AkeelZeki Mahdi].

2. EXPERIMENTAL PROCEDURE

The measurement occurred on girl under going from Polio in her left leg as a case study. The pathological subject is of age, weight, length and the residual palsy limb of 30years, 59kg, 156 cm, and 76 cm respectively.

so as to know the magnitude of stress that result from vibration in each position of knee Ankle Foot orthoses during dress up by patient must be measuring and calculate many necessary vibration data

1. Measure acceleration for KAFO by using portable acceleration sensor as shown in **Fig.1** in different position the sensor was located in

- a. The center of ankle.
- b. The mid of knee.
- c. The mid of the thigh

The sensor was recording data, while patient was walking in her normal speed on the ground in straight way.

2. Measuring pressure: The pressure between the leg and knee Ankle foot orthotic was measured by using piezoelectric sensor as shown in **Fig.2**. The pole of sensor connected with multi-meter devise to obtain the magnitude of voltage that result from response of sensor through the stance phase explained in **Fig.3**. The multi-meter and piezoelectric are interface with the computer and recording data as shown in **Fig. 4**. The pressure measured in shank and thigh region each position was divided into three parts longitudinal, in the middle and two parts on the terminal of the piece as shown in **Fig. 5**. The program of multi-meter giving maximum and minimum value of voltage with time Take maximum

value of this to be the pressure in this point .The procedure will be recurred to another part so as to obtain on the pressure values in all position that contact with muscles of leg as shown in **Fig. 6**. To convert the measurements to pressure, the following calibration steps must be done:

1. Using different disc masses of 50,100,...500gm
2. Put the gradual steps the masses over the pressure sensor and read the voltage from the multi-meter as shown
3. Develop the mass (or force) with voltage relationship.
4. Convert the force-voltage relation to pressure-voltage relation by dividing each force by the circular cross sectional area of sensor. The chart of calibration in **Fig. 7** is already now to using.

3. THE NUMEIRCAL ANLAYSIS

The finite element method (FEM) is now widely used in a variety of fields in engineering and science. Taking the advantage of the rapid development of digital computers with large memory capacity, as well as, fast computation. The method is recognized as one of the most powerful numerical methods because of its capabilities which include complex geometrical boundaries and non-linear material properties. In this work, FEM is with aid of ANSYS Workbench 14 software used as a numerical tool to determine the behavior of maximum stress [The Iron & Steel Society].

- Building the geometry as a model.
- Applying the boundary conditions load and obtaining the solution.
- Reviewing the results.

4. GRAPHING OF THE GEOMETRY

In this paper selection metal-metal knee ankle foot orthosis (KAFO) model was drawn by using CAD system (AUTOCAD) which processed according to an default pattern in three dimensions .The dimension was taken from the same KAFO that done on it measurement of experimental part. The aim of drawing models by AUTOCAD so as to use

in ANSYS workbench program for modeling, meshing and defining boundary condition such as applied load. The models is illustrate in the **Fig. 8**.

5. MATERIAL SELECTION IN KAFO'S

Various materials used in the manufacturing of KAFO depending on the type of KAFO such as (metal-metal, plastic-metal) and position of each section that building structure of KAFO like material used in the knee joint different from used in sole , the materials are: Stainless steel, Polypropylene and Aluminum

5.1. Stainless Steel

Stainless steel used for making the uprights side bars, knee joint system, ankle joint system and system shoe stirrup in of Metal – Metal KAFO as shown in **Fig. 8**. Stainless steel using in this sections because of the good characteristics like resistance to corrosion in many environments, their good mechanical properties over an extremely wide range of temperatures, and their superior resistance to oxidation and scaling at very high temperatures [The Iron & Steel Society], the mechanical properties of standard Stainless steel are listed in the **Table 2**.

5.2. Polypropylene

PP has low density and good flexibility and resistance to chemicals, abrasion and moisture, but decreased dimensional stability, mechanical strength, and resistance to UV (ultraviolet) light and heat [Maier C, & Calafut].Polypropylene used for making sole section of Metal – Metal KAFO, The mechanical properties of standard polypropylene are listed in the **Table 3**.

5.3. Aluminum:

Aluminum alloys 1200 were selected as the material for the calf band in the metal –metal KAFO in the thigh and shank section due to their light weight, high strength-to-weight ratio, corrosion resistance, and relatively low cost. , The mechanical properties of standard aluminum alloy 1200 are listed in the **Table 4**.

6. MESH OF THE MODEL

The meshing process has been done by choosing the volume, and then the shape of element was selected as tetrahedron (Automatic meshing), for metal-metal model as shown in **Fig. 9** contain total number of elements was (95298 elements) with total a number of nodes of (182108 nodes).

7. DEFINING THE ANALYSIS TYPE AND APPLYING LOAD

The boundary condition applying in the ANSYS Workbench software will be fixed support at the sole and the tip of the calf for both of shank and thigh segments. While, the interface pressure was distributed on the calf for both of shank and thigh segments shown in **Fig. 10**, and enters the values of acceleration measured in thigh, knee and Ankle shown in **Fig.11**.

8. RESULTS AND DISCUSSION

Observed by comparing the results of acceleration and frequency in the case of metal - metal knee ankle foot orthosis in **Table 1** to a person suffering from poliomyelitis disease find increase in the value of acceleration and frequency in knee joint this is due to more movement at knee joint comparing with ankle and thigh joints during gait walk cycle also it is very important to know that the knee joint is a bone region this is mean that there is no muscles working as a damper to reduce acceleration and frequency at this region according to this analysis acceleration and frequency increased in knee joint with about (62.3% and 46%) respectively comparing with ankle joint and with (30.19 %and 18.46%)respectively comparing with thigh joint.

All the readings of the interface pressure measurement are shown in **Fig. 5**.The difference between all these reading is related to value of pressure which is related to the activity of the muscles during which the pressure sensor is connected to measure the value of the interface pressure. The max value of IP in KAFO lower bands was recorded at gastrocnemius muscles with values of 28 Kpa While the max value of IP in KAFO

upper band is recorded at biceps femoris and semitendinosus muscles with values of 58 Kpa.

Figs. 12 to14 show the general contour of Von Mises stress for KAFO resulted from the ANSYS 14 program. The figures also shows that the Max value of stress is recorded at the upper medial KAFO side with 23.6 Mpa in region of bar contact with Gracilis muscle, while the Max. Value of stress at upper lateral KAFO side is 5.2 Mpa at the iliotibial band region. Also **Figs. 17 and 18** show that the Max value of stress is recorded at the lower lateral KAFO side with 2.49 Mpa in region of bar contact with soleus muscle, while the Max. value of stress at lower medial KAFO side is 1.62 Mpa at bar contact with peroneus longus muscle .**Figs.20-24** show the stress distribution in the uppers and lowers bands.it is clear that the Max. values of the stresses recorded in the upper bands is 6.5 Mpa exactly at the upper calf for thigh contact with semitendinosus muscles regions ,while the Max values of the stresses recorded in the lower band is 3.78 Mpa exactly at the upper calf for shank contact with gastrocnemius muscle regions .Results at figure(28) shows that the values of stresses at the upper KAFO bands (thigh) are greater than the lower KAFO bands (shank).This is certainly because of the values of pressure in the upper leg (thigh muscles)are higher than the pressure in shank muscles.

9. CONICLUSIONS

1- The max value of interface pressure in KAFO upper calf was recorded at semitendinosus muscles with values of 58 Kpa while the max value of interface pressure in KAFO lower calf was recorded at gastrocnemius muscles with values of 28Kpa.

2- The level of stresses in the upper KAFO thigh calf is greater than the lower KAFO shank thigh due to the high activities of the thigh muscles in comparison to shank muscles.

3- The maximum stress in the KAFO which was calculated numerically recorded at the bar contact with Gracilis muscle for Knee Ankle Foot Orthosis Aluminum.

4-Maximum value of frequency was recorded in knee joint with the value 6.5 Hz and maximum value of acceleration at knee joint about 5.578 m/s^2

Ankle	2.069	3.5
Knee	5.578	6.5
thigh	3.894	5.3

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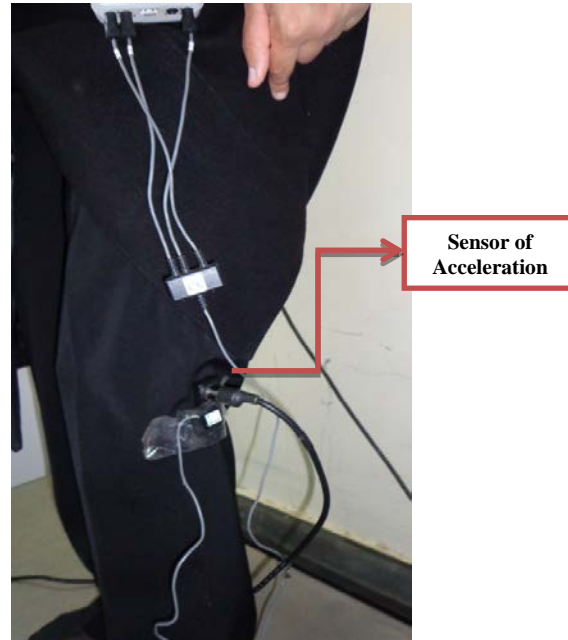


Figure 1. Acceleration sensor device.

Table1. Vibration data for metal KAFO.

Vibration data for the metal knee ankle foot orthosis		
Point	RMS Acceleration amplitude (m/s^2)	Frequency(Hz)
Ankle	2.069	3.5
Knee	5.578	6.5
thigh	3.894	5.3



Figure 2. The piezoelectric sensor.

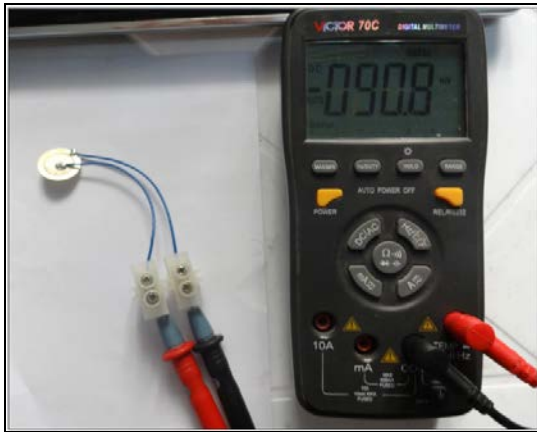


Figure 3. Pole of sensor connected with multi-meter device.



Figure 6. The pressure measured in shank region.

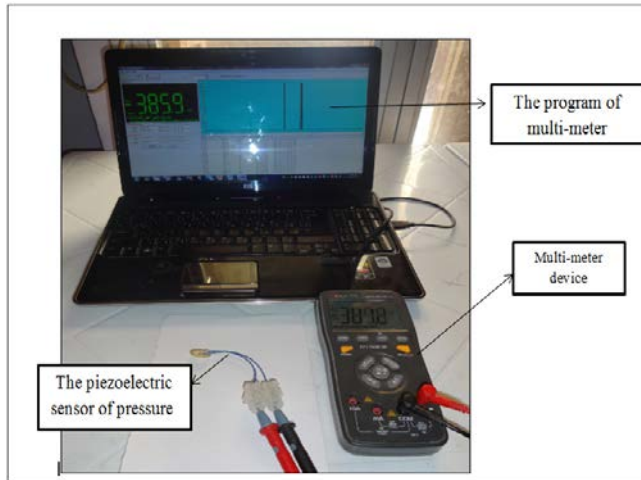


Figure 4. The multi-meter and piezoelectric are interface with the computer.

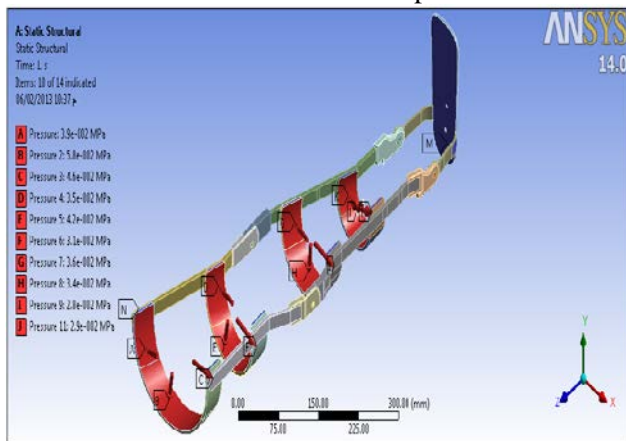


Figure 5. The model subjected to pressure load.

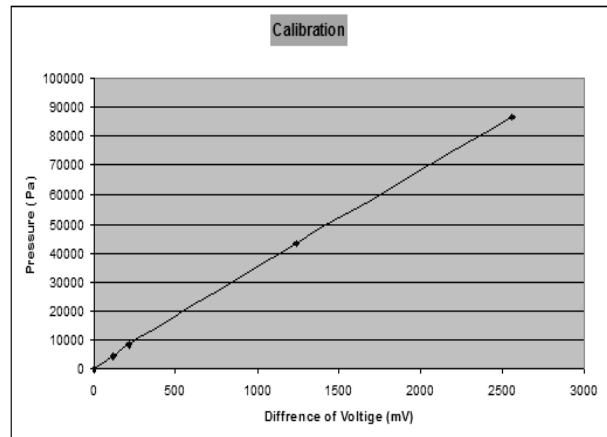


Figure 7. The calibration curve.

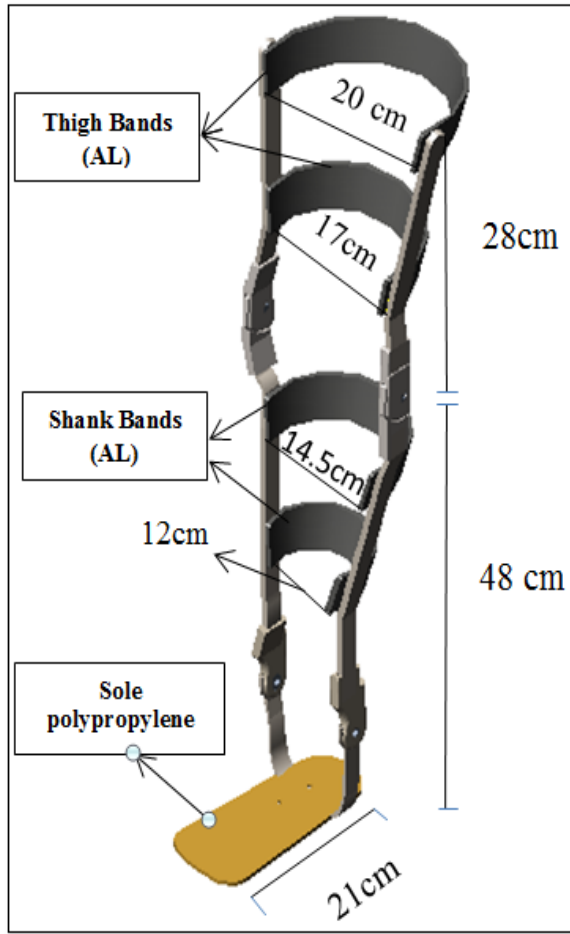


Figure 8. Graphing of the geometry and material selection in KAFO's.

Table 2. The mechanical properties of standard stainless steel.

Young's Modulus (GPa)	Poisson ratio	Density (kg/m^3)
210	0.3	7800

Table 3. The mechanical properties of standard polypropylene.

Young's Modulus (GPa)	Poisson ratio	Density (kg/m^3)
0.9	0.3	890

Table 4. The mechanical properties of aluminum alloy 1200.

Young's Modulus (GPa)	Poisson ratio	Density (kg/m^3)
70	0.3	2700

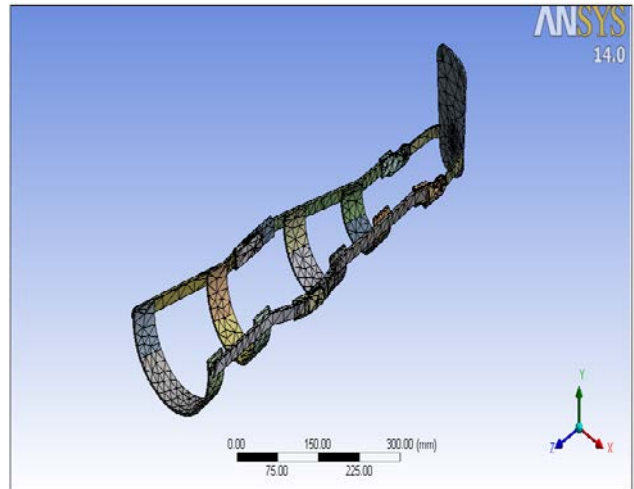


Figure 9. Mesh of the model.

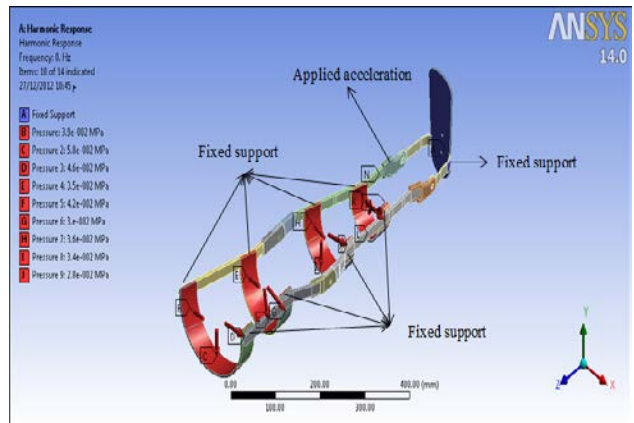


Figure 10. The model subjected to fixed support.

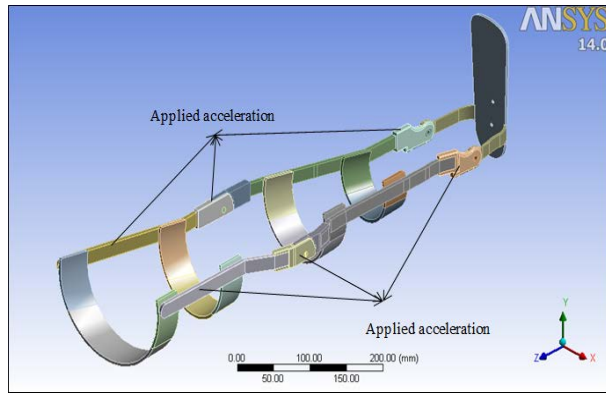


Figure 11. The model subjected to acceleration.

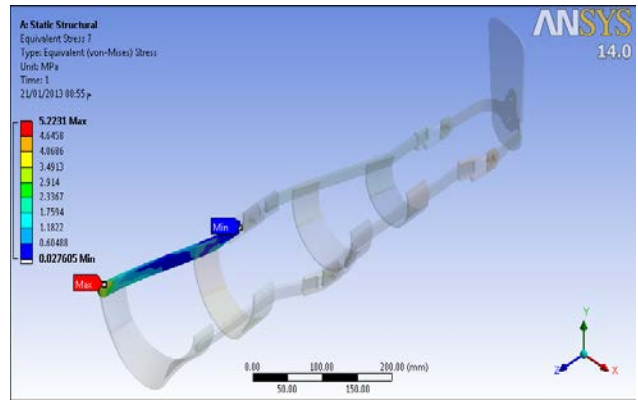


Figure 14. The Von-Misses stress due to loading boundary condition (IP) at bar contact with vastus lateralis muscle for KAFO Aluminum model.

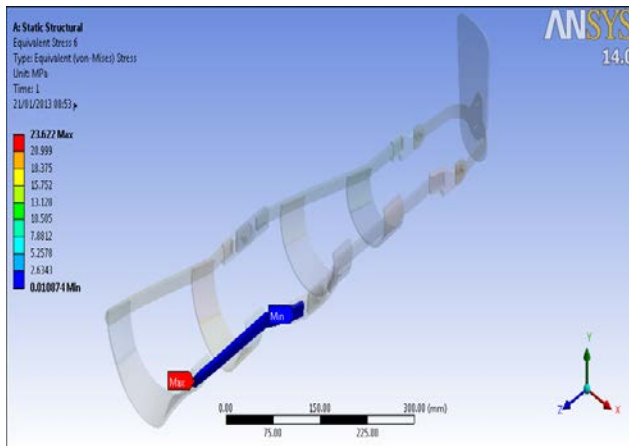


Figure 12. The Von-Misses stress due to loading boundary condition (IP) at bar contact with Gracilis muscle for KAFO Aluminum model.

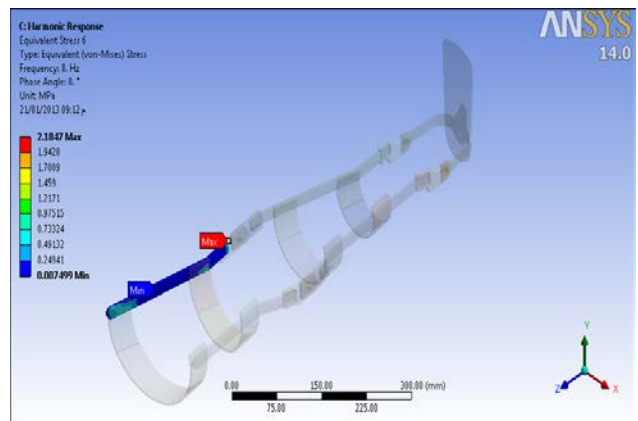


Figure 15. The Von-Misses stress due to harmonic body motion (acceleration) at bar contact with vastus lateralis muscle for KAFO Aluminum model

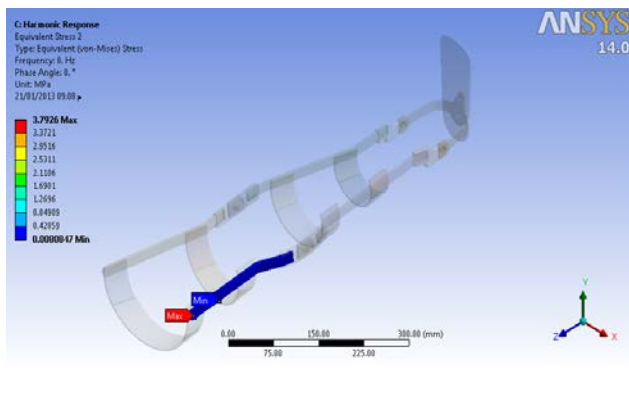


Figure 13. The Von-Misses stress due to harmonic body motion (acceleration) at bar contact with Gracilis muscle for KAFO Aluminum model.

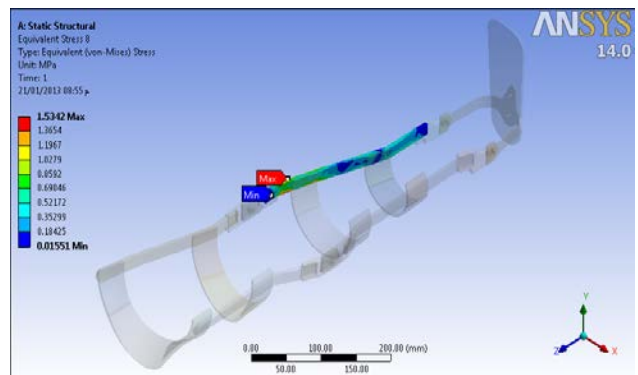


Figure 16. The Von-Misses stress due to loading boundary condition (IP) at bar contact with peroneus longus muscle for KAFO Aluminum model.

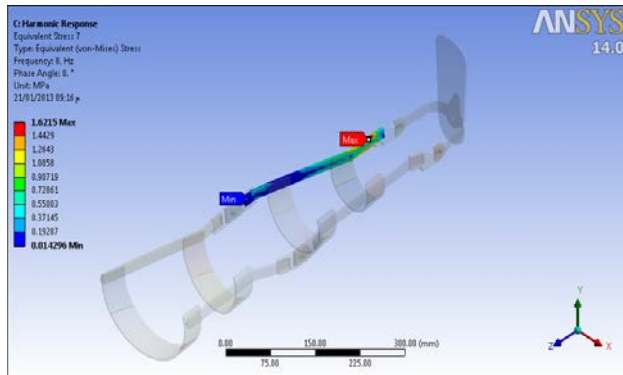


Figure 17. The Von-Misses stress due to harmonic body motion (acceleration) at bar contact with peroneus longus muscle for KAFO Aluminum model.

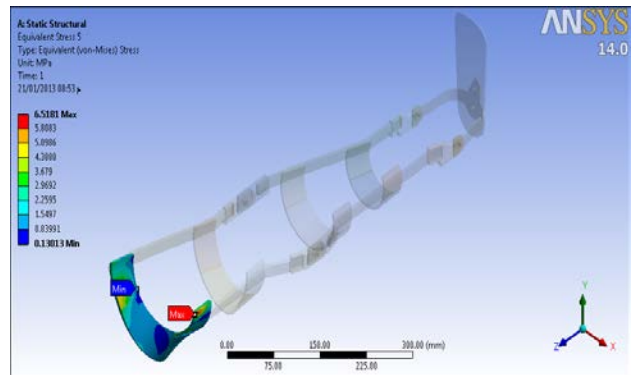


Figure 20. The Von-Misses stress due to loading boundary condition (IP) at upper calf for thigh contact with semitendinosus for KAFO.

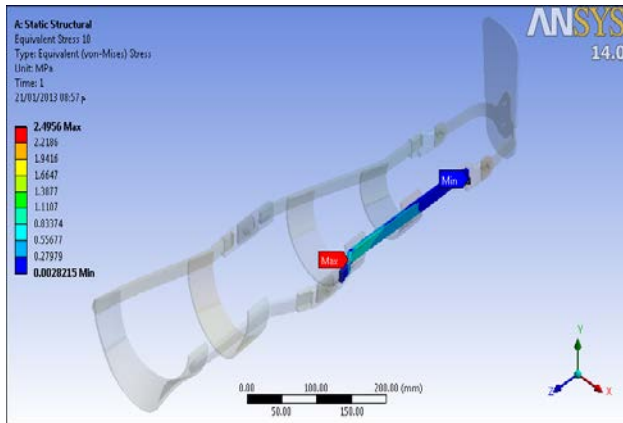


Figure 18. The Von-Misses stress due to loading boundary condition (IP) at bar contact with soleus muscle for KAFO Aluminum model.

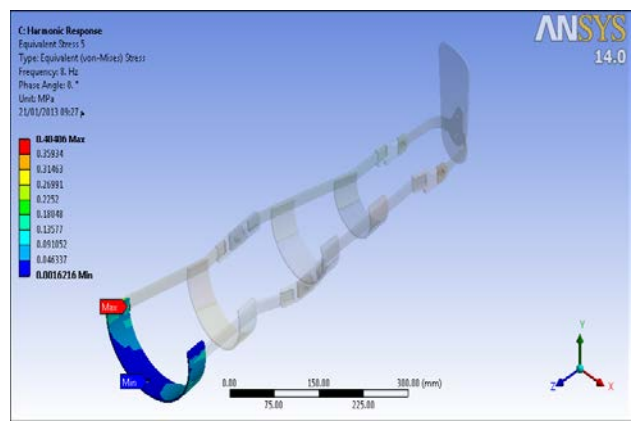


Figure 21. The Von-Misses stress due to harmonic body motion (acceleration) at upper calf for thigh contact with semitendinosus for KAFO.

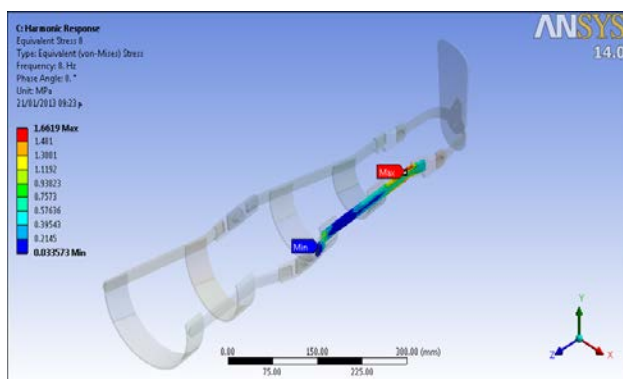


Figure 19. The Von-Misses stress due to harmonic body motion (acceleration) at bar contact with soleus muscle for KAFO Aluminum model.

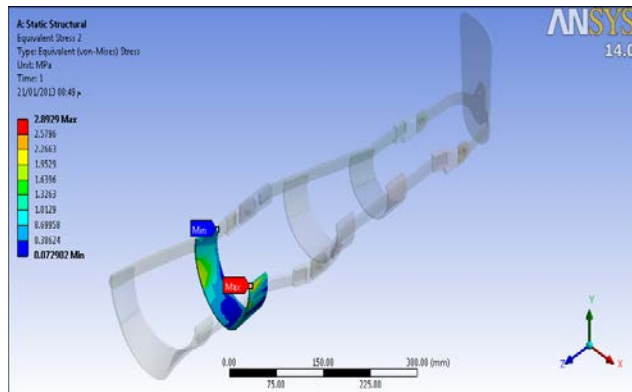


Figure 22. The Von-Misses stress due to loading boundary condition (IP) at lower calf for thigh contact with semitendinosus for KAFO.

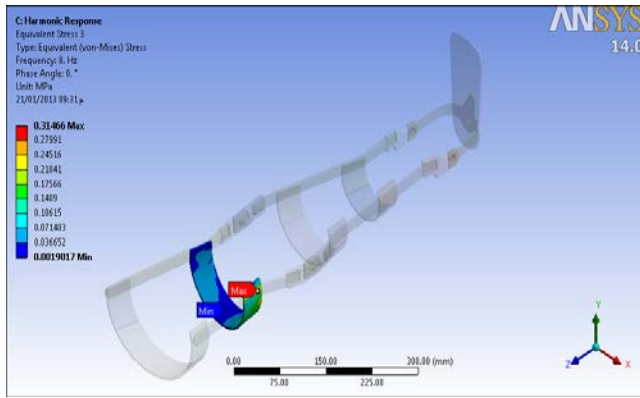


Figure 23 The Von-Misses stress due to harmonic body motion (acceleration) at lower calf for thigh contact with semitendinosus for KAFO.

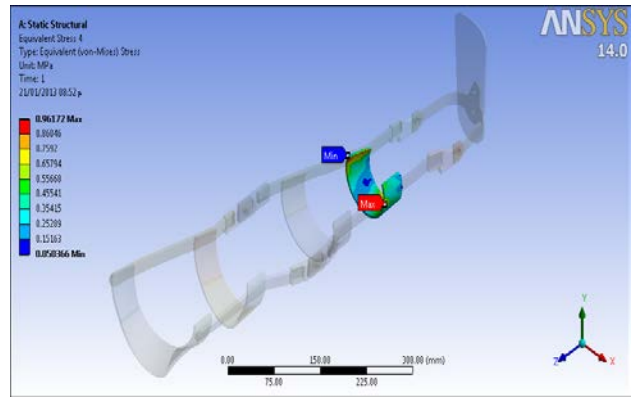


Figure 26. The Von-Misses stress due to loading boundary condition (IP) at lower calf for shank contact with gastrocnemius muscle for KAFO.

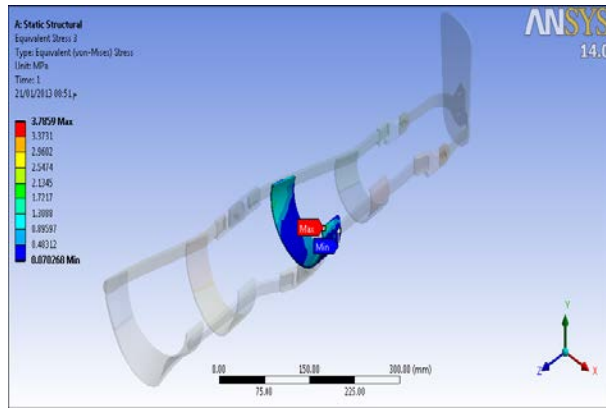


Figure 24. The Von-Misses stress due to loading boundary condition (IP) at upper calf for shank contact with gastrocnemius muscle for KAFO.

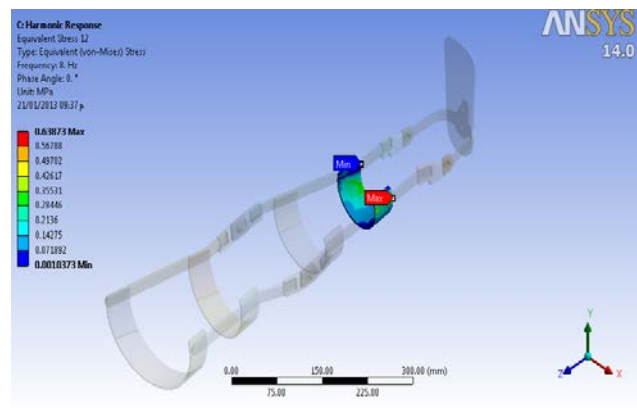


Figure 27. The Von-Misses stress due to harmonic body motion (acceleration) at lower calf for shank contact with gastrocnemius muscle for KAFO.

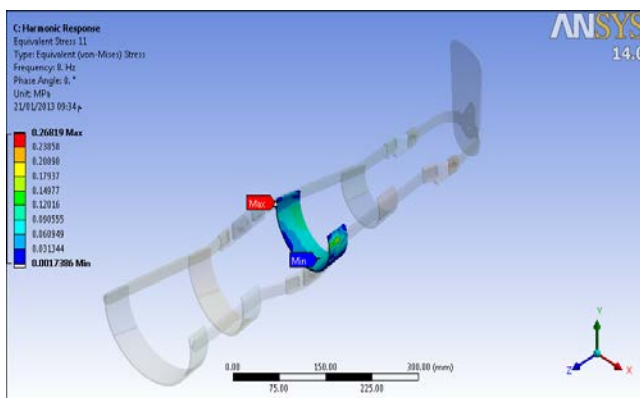


Figure 25. The Von-Misses stress due to harmonic body motion (acceleration) at upper calf for shank contact with gastrocnemius muscle for KAFO.

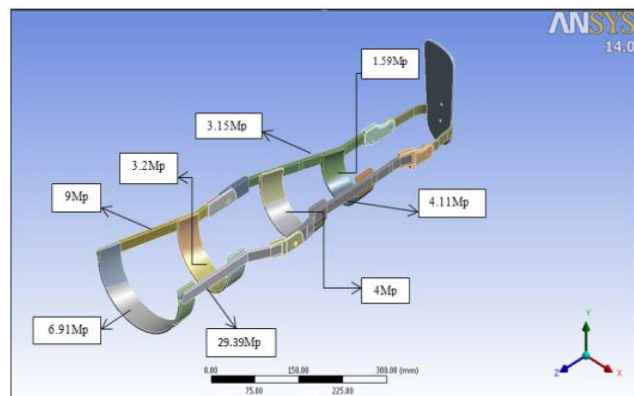


Figure 28. Total stress result from applied pressure and acceleration on the model.



FINITE ELEMENT EQUATION

The displacements are represented by the following equations:

$$\begin{aligned} u &= C_{11} + C_{12}X + C_{13}Y + C_{14}Z \\ v &= C_{21} + C_{22}X + C_{23}Y + C_{24}Z \\ w &= C_{31} + C_{32}X + C_{33}Y + C_{34}Z \end{aligned} \quad (1)$$

Considering the nodal displacements, the following equations must be satisfied:

$$\begin{aligned} u &= u_I \text{ at } X = X_I \text{ } Y = Y_I \text{ and } Z = Z_I \\ u &= u_J \text{ at } X = X_J \text{ } Y = Y_J \text{ and } Z = Z_J \\ u &= u_K \text{ at } X = X_K \text{ } Y = Y_K \text{ and } Z = Z_K \\ u &= u_L \text{ at } X = X_L \text{ } Y = Y_L \text{ and } Z = Z_L \end{aligned}$$

Similarly, requirements must be satisfied:

$$\begin{aligned} v &= v_I \text{ at } X = X_I \text{ } Y = Y_I \text{ and } Z = Z_I \\ v &= v_J \text{ at } X = X_J \text{ } Y = Y_J \text{ and } Z = Z_J \\ v &= v_K \text{ at } X = X_K \text{ } Y = Y_K \text{ and } Z = Z_K \\ v &= v_L \text{ at } X = X_L \text{ } Y = Y_L \text{ and } Z = Z_L \end{aligned}$$

Also:

$$\begin{aligned} w &= w_I \text{ at } X = X_I \text{ } Y = Y_I \text{ and } Z = Z_I \\ w &= w_J \text{ at } X = X_J \text{ } Y = Y_J \text{ and } Z = Z_J \\ w &= w_K \text{ at } X = X_K \text{ } Y = Y_K \text{ and } Z = Z_K \\ w &= w_L \text{ at } X = X_L \text{ } Y = Y_L \text{ and } Z = Z_L \end{aligned}$$

Substitution of respective nodal values into **Eq. (1)**, results in 12 equations and 12 unknowns:

$$\begin{aligned} u_I &= C_{11} + C_{12}X_I + C_{13}Y_I + C_{14}Z_I \\ u_J &= C_{11} + C_{12}X_J + C_{13}Y_J + C_{14}Z_J \\ u_K &= C_{11} + C_{12}X_K + C_{13}Y_K + C_{14}Z_K \\ u_L &= C_{11} + C_{12}X_L + C_{13}Y_L + C_{14}Z_L \end{aligned} \quad (2-a)$$

Also,

$$\begin{aligned} v_I &= C_{21} + C_{22}X_I + C_{23}Y_I + C_{24}Z_I \\ v_J &= C_{21} + C_{22}X_J + C_{23}Y_J + C_{24}Z_J \\ v_K &= C_{21} + C_{22}X_K + C_{23}Y_K + C_{24}Z_K \\ v_L &= C_{21} + C_{22}X_L + C_{23}Y_L + C_{24}Z_L \end{aligned} \quad (2-b)$$

And

$$\begin{aligned} w_I &= C_{31} + C_{32}X_I + C_{33}Y_I + C_{34}Z_I \\ w_J &= C_{31} + C_{32}X_J + C_{33}Y_J + C_{34}Z_J \\ w_K &= C_{31} + C_{32}X_K + C_{33}Y_K + C_{34}Z_K \\ w_L &= C_{31} + C_{32}X_L + C_{33}Y_L + C_{34}Z_L \end{aligned} \quad (2-c)$$

Solving for the unknown C-coefficients, substituting the results back into **Eq.(1)**, and regrouping the parameters, obtained:

$$\begin{aligned} u &= S_1u_I + S_2u_J + S_3u_K + S_4u_L \\ v &= S_1v_I + S_2v_J + S_3v_K + S_4v_L \\ w &= S_1w_I + S_2w_J + S_3w_K + S_4w_L \end{aligned} \quad (3)$$

The shape functions are:

$$\begin{aligned} S_1 &= \frac{1}{6V} (a_I + b_I X + c_I Y + d_I Z) \\ S_2 &= \frac{1}{6V} (a_J + b_J X + c_J Y + d_J Z) \\ S_3 &= \frac{1}{6V} (a_K + b_K X + c_K Y + d_K Z) \\ S_4 &= \frac{1}{6V} (a_L + b_L X + c_L Y + d_L Z) \end{aligned} \quad (4)$$

Where V, the volume of the tetrahedral element, is computed from:

$$6V = \det \begin{vmatrix} 1 & X_I & Y_I & Z_I \\ 1 & X_J & Y_J & Z_J \\ 1 & X_K & Y_K & Z_K \\ 1 & X_L & Y_L & Z_L \end{vmatrix} \quad (5)$$

The a_i, b_i, c_i, d_i... d_L-terms are:

$$\begin{aligned} a_I &= \det \begin{vmatrix} X_J & Y_J & Z_J \\ X_K & Y_K & Z_K \\ X_L & Y_L & Z_L \end{vmatrix} & b_I &= -\det \begin{vmatrix} 1 & Y_J & Z_J \\ 1 & Y_K & Z_K \\ 1 & Y_L & Z_L \end{vmatrix} \\ c_I &= \det \begin{vmatrix} X_J & 1 & Z_J \\ X_K & 1 & Z_K \\ X_L & 1 & Z_L \end{vmatrix} & d_I &= -\det \begin{vmatrix} X_J & Y_J & 1 \\ X_K & Y_K & 1 \\ X_L & Y_L & 1 \end{vmatrix} \end{aligned} \quad (6)$$

The only six independent stress components are needed to characterize the general state of stress at a point. These components are:

$$[\sigma]^T = [\sigma_{xx} \ \sigma_{yy} \ \sigma_{zz} \ \tau_{xy} \ \tau_{yz} \ \tau_{xz}] \quad (7)$$

Where σ_{xx} , σ_{yy} , and σ_{zz} are the normal stresses and τ_{xy} , τ_{yz} , and τ_{xz} are the shear-stress components. For the displacement vectors that measure the changes occurring in the position of a point within a body when the body is subjected to a load, it may be recall

that the displacement vector $\vec{\delta}$ can be written in terms of its Cartesian component as:

$$\vec{\delta} = u(x, y, z) \vec{i} + v(x, y, z) \vec{j} + w(x, y, z) \vec{k} \quad (8)$$

The general state of strain is characterized by six independent components as given by:

$$[\varepsilon]^T = [\varepsilon_{xx} \ \varepsilon_{yy} \ \varepsilon_{zz} \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{xz}] \quad (9)$$

Where ε_{xx} , ε_{yy} , and ε_{zz} are the normal strains and γ_{xy} , γ_{yz} , and γ_{xz} are the shear-strain components. The relationship between the strain and the displacement is represented by:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} & \varepsilon_{yy} &= \frac{\partial v}{\partial y} & \varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned} \quad (10)$$

Equations (10) can be represented in matrix form as:

$$\{\varepsilon\} = LU \quad (11)$$

Where:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad \text{and} \quad LU = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{Bmatrix}$$

L is commonly referred to as the linear-differential operator.

Over the elastic region of material, there is also exists a relationship between the state of stressed and strains, according to the generalized Hook's law. This relationship is given by the following equations:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \\ \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \\ \gamma_{xy} &= \frac{1}{G} \tau_{xy} & \gamma_{yz} &= \frac{1}{G} \tau_{yz} & \gamma_{xz} &= \frac{1}{G} \tau_{xz} \end{aligned} \quad (12)$$

The relationship between the stress and strain can be expressed in a compact-matrix form as:

$$\{\sigma\} = [D]\{\varepsilon\} \quad (13)$$

Where,

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

$$[D] = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

For a solid material under triaxial loading, the strain energy Λ is:

$$\Lambda^{(e)} = \frac{1}{2} \int_V (\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz}) dV \quad (14)$$

Or, in a compact matrix form,

$$\Lambda^{(e)} = \frac{1}{2} \int_V [\sigma]^T \{\varepsilon\} dV \quad (15.a)$$

Substituting for stresses in terms of strains using Hook's law, equation (14) can be written as:

$$\Lambda^{(e)} = \frac{1}{2} \int_V \{\varepsilon\}^T [D] \{\varepsilon\} dV \quad (15.b)$$

By using the four-node tetrahedral element to formulate the stiffness matrix. Recall that this element has four nodes, with each node having three translational degrees of freedom in the nodal x-, y-, and z-directions. The displacements u, v, and w in terms of the nodal values and are represented by:



$$\{u\} = [S]\{U\} \tag{16}$$

Where,

$$\{u\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$[S] = \begin{bmatrix} S_1 & 0 & 0 & S_2 & 0 & 0 & S_3 & 0 & 0 & S_4 & 0 & 0 \\ 0 & S_1 & 0 & 0 & S_2 & 0 & 0 & S_3 & 0 & 0 & S_4 & 0 \\ 0 & 0 & S_1 & 0 & 0 & S_2 & 0 & 0 & S_3 & 0 & 0 & S_4 \end{bmatrix}$$

$$\{U\} = \begin{Bmatrix} u_I \\ v_I \\ w_I \\ u_J \\ v_J \\ w_J \\ u_K \\ v_K \\ w_K \\ u_L \\ v_L \\ w_L \end{Bmatrix}$$

Beginning by relating the strains to the displacement field and, in turn, to the nodal displacements through the shape functions. It is needed to take the derivatives of the components of the displacement field with respect to the x-, y-, and z- coordinates according to the strain-displacement relation, Eq.(11). The operation results in:

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial S_1}{\partial x} & 0 & 0 & \frac{\partial S_2}{\partial x} & 0 & 0 & \frac{\partial S_3}{\partial x} & 0 & 0 & \frac{\partial S_4}{\partial x} & 0 & 0 \\ 0 & \frac{\partial S_1}{\partial y} & 0 & 0 & \frac{\partial S_2}{\partial y} & 0 & 0 & \frac{\partial S_3}{\partial y} & 0 & 0 & \frac{\partial S_4}{\partial y} & 0 \\ 0 & 0 & \frac{\partial S_1}{\partial z} & 0 & 0 & \frac{\partial S_2}{\partial z} & 0 & 0 & \frac{\partial S_3}{\partial z} & 0 & 0 & \frac{\partial S_4}{\partial z} \\ \frac{\partial S_1}{\partial y} & \frac{\partial S_1}{\partial x} & 0 & \frac{\partial S_2}{\partial y} & \frac{\partial S_2}{\partial x} & 0 & \frac{\partial S_3}{\partial y} & \frac{\partial S_3}{\partial x} & 0 & \frac{\partial S_4}{\partial y} & \frac{\partial S_4}{\partial x} & 0 \\ 0 & \frac{\partial S_1}{\partial z} & \frac{\partial S_1}{\partial y} & 0 & \frac{\partial S_2}{\partial z} & \frac{\partial S_2}{\partial y} & 0 & \frac{\partial S_3}{\partial z} & \frac{\partial S_3}{\partial y} & 0 & \frac{\partial S_4}{\partial z} & \frac{\partial S_4}{\partial y} \\ \frac{\partial S_1}{\partial z} & 0 & \frac{\partial S_1}{\partial x} & \frac{\partial S_2}{\partial z} & 0 & \frac{\partial S_2}{\partial x} & \frac{\partial S_3}{\partial z} & 0 & \frac{\partial S_3}{\partial x} & \frac{\partial S_4}{\partial z} & 0 & \frac{\partial S_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_I \\ v_I \\ w_I \\ u_J \\ v_J \\ w_J \\ u_K \\ v_K \\ w_K \\ u_L \\ v_L \\ w_L \end{Bmatrix} \tag{17}$$

Substituting for the shape function using the relation of equation (4) and differentiating:

$$\{\epsilon\} = [B]\{u\} \tag{18}$$

Where,

$$[B] = \frac{1}{6V} \begin{bmatrix} b_I & 0 & 0 & b_J & 0 & 0 & b_K & 0 & 0 & b_L & 0 & 0 \\ 0 & c_I & 0 & 0 & c_J & 0 & 0 & c_K & 0 & 0 & c_L & 0 \\ 0 & 0 & d_I & 0 & 0 & d_J & 0 & 0 & d_K & 0 & 0 & d_L \\ c_I & b_I & 0 & c_J & b_J & 0 & c_K & b_K & 0 & c_L & b_L & 0 \\ 0 & d_I & c_I & 0 & d_J & c_J & 0 & d_K & c_K & 0 & d_L & c_L \\ d_I & 0 & b_I & d_J & 0 & b_J & d_K & 0 & b_K & d_L & 0 & b_L \end{bmatrix}$$

And the volume V and the b-, c-, and d- terms are given by Eqs. (5) and (6). Substituting into the strain energy equation for the strain components in terms of the displacements, obtained:

$$\Lambda^{(e)} = \frac{1}{2} \int_V \{\epsilon\}^T [D] \{\epsilon\} dV = \frac{1}{2} \int_V [U]^T [B]^T [D] [B] [U] dV \tag{19}$$

Differentiating with respect to the nodal displacements yields:

$$\frac{\partial \Lambda^{(e)}}{\partial U_K} = \frac{\partial}{\partial U_K} \left(\frac{1}{2} \int_V [U]^T [B]^T [D] [B] [U] dV \right) \tag{20}$$

for K=1, 2,.., 12

Evaluating of Eq.(20) results in the expression

$[K]^{(e)} \{U\}$ and, subsequently, the expression for the stiffness matrix, which is:

$$[K]^{(e)} = \int_V [B]^T [D] [B] dV = V [B]^T [D] [B] \tag{21}$$

Where V is the volume of the element. Note that the resulting stiffness matrix will have the dimensions of 12x12.

The load matrix for a tetrahedral element is a 12x1 matrix. For a concentrated – loading situation, the load matrix is formed by placing the components of the load at appropriate nodes in a appropriate directions. For a distributed load, the load matrix is computed from the equation:

$$\{F\}^{(e)} = \int_A [S]^T \{P\} dA \tag{22}$$

Where $\{P\} = \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}$

And A represents the surface over which the distributed-load components are acting. The surfaces

of the tetrahedral element are triangular in shape. Assuming that the distributed load acts on the I-J-K surface, the load matrix becomes:

$$\{F\}^{(e)} = \frac{A_{I-J-K}}{3} \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (23)$$

The load matrix for distributed load acting on the other surface of the tetrahedral element is obtained in a similar fashion.