



ANALYTICAL SOLUTION OF UNSATURATED SOIL WATER FLOW FROM A POINT SOURCE

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ABSTRACT

Water flow into unsaturated porous media is governed by the Richards' partial differential equation expressing the mass conservation and Darcy's laws. The Richards' equation may be written in three forms, where the dependent variable is pressure head or moisture content, and the constitutive relationships between water content and pressure head allow for conversion of one form into the other. In the present paper, the "moisture-based" form of Richards' equation is linearized by applying Kirchhoff's transformation, which combines the soil water diffusivity and soil water content. Then the similarity method is used to obtain the analytical solution of wetting front position. This exact solution is obtained by means of Lie's method of infinitesimal transformation groups. The predicted results of the analytical solution agreed well with available results of experiments and numerical solutions.

الخلاصة

جرى خلال العقد الاخير تقدم كبير في تصميم و ادارة نظام الري بالتنقيط ، و بضمنه دراسة و تحليل حركة الماء داخل التربة من مصدر نقطي.. تم وصف انتشار الماء من مصدر نقطي على سطح التربة في وسط متماثل و متجانس بمعادلة ريتشاردز (Richards' Equation) التفاضلية الجزئية ، التي تربط بين قانوني حفظ الكتلة و الطاقة، و قد تكتب هذه المعادلة بثلاث صيغ قياسية وهي اما على أساس الضغط، او على أساس الرطوبة، او على أساس مختلط. قد تم في هذا البحث ايجاد حلاً تحليلياً لمعادلة ريتشاردز الموصوفة ، بصيغة اساسها الرطوبة ، باستخدام بعض الفرضيات منها ان التربة متجانسة و متماثلة الخواص الفيزيائية، و عدم حدوث تبخر من سطح التربة، و المحتوى الرطوبي الابتدائي خلال التربة منتظم، و عدم حدوث تداخل بين انماط الرطوبة من المنقطات المنفردة، و كون تأثيرات الجاذبية قابلة للاهمال، اي ان الجريان يحدث في تربة ناعمة أو متوسطة النسجة. كما تم أيضا افتراض كون الجريان متماثل محوريا (على طول المحور الشاقولي)، و بذلك يمكن تحديد المحتوى المائي الحجمي بدلالة المسافة من المصدر النقطي و الوقت. قد تم استخدام تحويل كيرشوف (Kirchhoff's transformation)، الذي يدمج أنتشارية ماء التربة و المحتوى الرطوبي للتربة، لتحويل المعادلة التفاضلية الجزئية غير الخطية إلى معادلة تفاضلية جزئية خطية، و من ثم استخدمت طريقة التماثل (التشابه) لتحويل المعادلة التفاضلية الجزئية الخطية إلى معادلة تفاضلية اعتيادية من اجل الحصول على الحل التحليلي.

قد تم التحقق من نتائج الحل التحليلي للنموذج الرياضي باستخدام بيانات مقياسه مختبريا"، و نتائج حلول عددية لدراسات سابقة، و قد تبين ان هناك توافق جيد بين نتائج الحل التحليلي و القياسات المختبرية و القيم المحسوبة من الحل العددي في المراحل المبكرة من عملية الارتشاح ، بعد هذه الفترة تبدأ قيم الحل النظري بالابتعاد عن مثيلاتها للبحوث العملية و الحلول العددية السابقة ، و ذلك بسبب إهمال تأثير الجاذبية. ان الحل الذي تم الحصول عليه مفيد للغاية عند تصميم منظومات الري بالتنقيط ، و ذلك لانه يوفر وسيلة لمعرفة حجم التربة المبتلة بدلالة الزمن ، و بذلك يمكن ان تساهم النتائج في تحديد فواصل المنقطات أو وقت الارواء .

KEYWORDS: Unsaturated flow, Richards' equation, trickle irrigation, point source, Kirchhoff's transformation, similarity.

INTRODUCTION

Trickle irrigation is the most common micro-irrigation method based on supplying water close to the rooted soil volume. This irrigation method allows the effective wetted soil volume to be reduced, thus, evaporation and deep percolation (water and nutrients) to be limited. Trickle irrigation management requires prediction of the wetted soil volume because an over application of water results in loss of both water and fertilizers beyond the root zone. Information on moisture distribution patterns under point source trickle emitters is a pre-requisite for the design and operation of trickle – irrigation systems. The distribution pattern is influenced by the soil properties and the manner water is applied and withdrawn from the soil profile. Flow from a point – source trickle emitter, because of its multi-dimensional nature and high frequency of water application, leads to complexities in modeling soil moisture dynamics (Narda and Lubana, 2001). Wetting pattern from trickle source can be obtained by either direct measurement of soil wetting in field, which is site - specific, or by simulation using some numerical or analytical models. In most of models, the Richards' equation governing water flow under unsaturated flow conditions have been used to simulate soil water matrix potential or water content distribution in wetted soil. Also the hydraulic conductivity in unsaturated flow equations is highly nonlinear and show high spatial variable. Numerical and analytical methods have been used to solve unsaturated flow equations (Battam, et al., 2003).

Several studies have been conducted to determine the flow pattern from trickle sources (Bresler, et al., 1971; Clothier and Scotter, 1982; Hachum, et al., 1976). Mathematical models have been developed to analyze multidimensional infiltration under trickle sources by using non-linear water flow equation (Brandt, et al., 1971; Taghavi, et al., 1984; Lafolie, et al., 1989). Analytical solutions for the corresponding linearized form of the water flow equation with or without plant uptake have been developed for steady-state conditions (Philip, 1971; Raats, 1971). The time-dependent infiltration equation for surface sources of water with various two and three-dimensional geometries has been treated by Warrick (1974) and Lomen and Warrick (1976) by using a linearization approach. By using the unsteady, linearized solution, the wetting-front locations during infiltration were determined by

considering the advance of parabolas of constant matric flux potential with time. Clothier and Scotter (1982) and Clothier, et al., (1985) studied infiltration from a hemispherical cavity by using simple absorption theory. They developed an analytical solution for three-dimensional infiltration with an assumption that, at early times, the gravity term in the flow equation is insignificant relative to the sorption term. Chung (1987) studied a three-dimensional infiltration from a point source by applying an alternating direction implicit (ADI) finite difference method. He used a Darcy-law based soil water equation with a cylindrical coordinate system and a constant flow rate at the point source. Two analytical techniques for studying infiltration from a surface point source have been proposed by Ben-Asher, et al., (1986) and Healy and Warrick, (1988). Analytical models provide a rapid method for determining the wetting front position (Revol, et al., 1997; Cook, et al., 2003). These models are based on the assumption of a point source and certain forms for the physical properties of soil and water content distributions (Philip, 1984; Revol, et al., 1997).

The analysis presented here is simplified by the assumptions of homogeneous and isotropic soils, no evaporation from the soil surface, uniform initial moisture contents, no overlapping between moisture patterns from point sources, and gravity effects are negligible, i.e., a case of flow in medium or fine-textured soils is considered. The technique avoids other limitations of analytical methods such as steady flow.

WATER FLOW EQUATION

The soil-water flow equation can be developed by combining the continuity equation and Darcy's equation. The governing equation for the soil - water flow can be expressed as (Philip, 1969):

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [K(\theta) \nabla \psi(\theta)] \quad (1)$$

where

θ : volumetric soil water content which is a function of location and time, L^3/L^3 ,

$K(\theta)$: unsaturated soil hydraulic conductivity which is a function of volumetric soil water content, L/T ,



$\psi(\theta)$: total water head which is a function of the volumetric soil water content, L ,

t : time, T ,

∇ : the del operator (gradient operator), and

$\nabla \cdot$: the divergence operator.

It is usually convenient to separate the total potential into gravitational and matric potentials. Such a separation yields the general form of Richards' equation, or:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [K(\theta) \nabla h(\theta)] - \frac{\partial K(\theta)}{\partial z} \quad (2)$$

where

h : matric potential head, L , and

z : gravitational potential head expressed as the depth below soil surface (positive downward), L .

The infiltration phenomenon from a surface point source into a homogeneous and isotropic soil of a uniform initial volumetric water content, θ_i , can be described by the equation governing moisture flow in an unsaturated soil. The moisture-based form is (Philip, 1969):

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [D(\theta) \nabla \theta] - \frac{\partial K(\theta)}{\partial z} \quad (3)$$

where

$D(\theta)$ = unsaturated soil water diffusivity, L^2/T .

At short times during three-dimensional infiltration and in medium or fine-textured soils the gravity term in eq. (3) is insignificant relative to the sorption term, so the infiltration process is approximately absorption (gravity-free) with radial symmetry (Clothier and Scotter, 1982). Therefore, eq. (3) reduces to the nonlinear diffusion equation (Philip, 1969):

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [D(\theta) \nabla \theta] \quad (4)$$

In the present study, consider a systems exhibiting spherical radial symmetry, then the volumetric

water content, θ , can be expressed in terms of the radial distance from the source, r , and the time, t . Thus, eq. (4) can be written as:

$$\frac{\partial \theta}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(D(\theta) r^2 \frac{\partial \theta}{\partial r} \right) \quad (5)$$

The initial and boundary conditions are (as shown in Fig. 1):

$$\begin{aligned} \theta &= \theta_i, \text{ at } t=0, \quad r > 0 \\ \theta &= \theta_s, \text{ at } t > 0, \quad r = 0, \text{ and} \\ \theta &= \theta_i, \text{ at } t > 0, \quad r = \infty \end{aligned} \quad (6)$$

where

r = radial distance from the source, L ,

θ_i = initial soil water content, L^3/L^3 , and

θ_s = saturated soil water content, L^3/L^3 .

MODEL DEVELOPMENT

A similarity substitution usefully reduces the number of independent variable in a partial differential equation only when the variables removed from the equation are removed also from all the governing conditions by the same substitution. A similarity solution of a partial differential equation is obtained by first transforming it into an ordinary differential equation. The equation has been solved by two methods one solution utilized the Boltzmann's transformation and the other utilized similarity techniques. The complete solution is described in detail elsewhere by Abid, (2006); the following is a brief description.

1. Boltzmann Similarity Transformation

The form of eq. (5) can be modified by the application of Kirchhoff's integral transformation in which the dependent variable θ is transformed into a new variable ϕ by means of:

$$\phi = \int_{\theta_i}^{\theta} D(\theta) \, d\theta \quad (7)$$

where

ϕ = matric flux potential, L^2/T .

Application of Kirchhoff's integral transformation to eq. (5) yields:

$$\frac{1}{D_*} \frac{\partial \phi}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial \phi}{\partial r} \right] \quad (8)$$

where

D_* = soil water diffusivity of linearized model, L^2/T .

which will be subjected to the following conditions:

$$\left. \begin{aligned} \phi &= 0, \quad \text{at } t=0, \quad r > 0 \\ \phi &= \mu_i, \quad \text{at } t > 0, \quad r = 0, \text{ and} \\ \phi &= 0, \quad \text{at } t > 0, \quad r = \infty \end{aligned} \right\} \quad (9)$$

where

$$\mu_i = \int_{\theta_i}^{\theta_s} D(\theta) d\theta \quad (10)$$

To eliminate the r^2 term in eq. (8), the following substitution can be used:

$$\phi = \frac{\lambda(r, t)}{r} \quad (11)$$

Therefore, eq. (8) reduces to:

$$\frac{\partial^2 \lambda}{\partial r^2} = \frac{1}{D_*} \frac{\partial \lambda}{\partial t} \quad (12)$$

which is now subjected to the conditions:

$$\left. \begin{aligned} \lambda &= 0, \quad \text{at } t=0, \quad r > 0 \\ \lambda &= r \mu_i, \quad \text{at } t > 0, \quad r = 0, \text{ and} \\ \lambda &= 0, \quad \text{at } t > 0, \quad r = \infty \end{aligned} \right\} \quad (13)$$

Eq. (12) may be transformed into an ordinary differential equation by applying the well-known Boltzmann's transformation (Boltzmann, 1894, this has been cited in Remson, et al., 1971). Such a transformation is defined by:

$$\left. \begin{aligned} \rho &= \frac{r}{2 \sqrt{D_* t}} \\ \lambda(\rho) &= \lambda(r, t) \end{aligned} \right\} \quad (14)$$

where

ρ = the similarity variable, and

$$D_* = \frac{\mu_i}{(\theta_s - \theta_i)} = \frac{1}{(\theta_s - \theta_i)} \int_{\theta_i}^{\theta_s} D(\theta) d\theta \quad (15)$$

Thus, application of the similarity transformation, defined by eq. (14) to eq. (12) yields the ordinary differential equation:

$$\frac{d^2 \lambda}{d \rho^2} + 2\rho \frac{d \lambda}{d \rho} = 0 \quad (16)$$

subjected to:

$$\left. \begin{aligned} \lambda(0) &= r \mu_i, \text{ and} \\ \lambda(\infty) &= 0 \end{aligned} \right\} \quad (17)$$

After separating the variables, integrating, and rearranging, the resulting equation is:

$$\int_{\theta_i}^{\theta} D(\theta) d\theta = \mu_i \operatorname{erfc}(\rho) \quad (18)$$

Substituting μ_i for $\int_{\theta_i}^{\theta_s} D(\theta) d\theta$ gives the following:

$$\int_{\theta_i}^{\theta} D(\theta) d\theta = \operatorname{erfc}(\rho) \int_{\theta_i}^{\theta_s} D(\theta) d\theta \quad (19)$$

The analytical solution to the water flow equation for some cases of soil water diffusivity is shown in **Table 1**.

2. Classical Similarity Reductions

The classical method for finding symmetry reductions of partial differential equations is the Lie- group method of infinitesimal transformations (Ames, 1967; and Bluman and Cole, 1974; Ovsianikov, 1982; Bluman and Kumei, 1989; Olver, 1993). The method of solution depends on the application of one-parameter group transformation to the partial differential equation [eq. (12)]. Under this transformation the two independent variables will be reduced by one, and a differential equation in only one independent variable is obtained, which is the similarity variable.

To apply the classical method to the second order partial differential eq. (12). A one parameter group of infinite infinitesimal transformations is sought which takes the (r, t, λ) - space into itself and under which eq. (12) is invariant, i. e.:

$$\left. \begin{aligned} \bar{r} &= r + \varepsilon \xi(r, t, \lambda) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon \tau(r, t, \lambda) + O(\varepsilon^2), \text{ and} \\ \bar{\lambda} &= \lambda + \varepsilon \eta(r, t, \lambda) + O(\varepsilon^2). \end{aligned} \right\} \quad (20)$$

where

ε = group parameter, and

$O(\varepsilon^2)$ = order of the parameter, ε .

Also the derivatives of λ are transformed according to:

$$\left. \begin{aligned} \bar{\lambda}_{\bar{t}} &= \lambda_t + \varepsilon [\tilde{\eta}_t] + O(\varepsilon^2), \text{ and} \\ \bar{\lambda}_{\bar{r}\bar{r}} &= \lambda_{rr} + \varepsilon [\tilde{\eta}_{rr}] + O(\varepsilon^2). \end{aligned} \right\} \quad (21)$$

where

$[\tilde{\eta}_t]$, and $[\tilde{\eta}_{rr}]$ = infinitesimals for transformations of the derivatives η_t and η_{rr} , respectively.

and

$$\begin{aligned} [\tilde{\eta}_t] &= \frac{\partial \eta}{\partial t} + \left[\frac{\partial \eta}{\partial \lambda} - \frac{\partial \tau}{\partial t} \right] \lambda_t - \frac{\partial \xi}{\partial t} \lambda_r \\ &\quad - \frac{\partial \tau}{\partial \lambda} \lambda_t^2 - \frac{\partial \xi}{\partial \lambda} \lambda_r \lambda_t. \end{aligned} \quad (22)$$

$$\begin{aligned} [\tilde{\eta}_{rr}] &= \frac{\partial^2 \eta}{\partial r^2} + \left(2 \frac{\partial^2 \eta}{\partial r \partial \lambda} - \frac{\partial^2 \xi}{\partial r^2} \right) \lambda_r \\ &\quad - \frac{\partial^2 \tau}{\partial r^2} \lambda_t + \left(\frac{\partial^2 \eta}{\partial \lambda^2} - 2 \frac{\partial^2 \xi}{\partial r \partial \lambda} \right) \lambda_r^2 \\ &\quad - 2 \frac{\partial^2 \tau}{\partial r \partial \lambda} \lambda_r \lambda_t - \frac{\partial^2 \xi}{\partial \lambda^2} \lambda_r^3 \\ &\quad - \frac{\partial^2 \tau}{\partial \lambda^2} \lambda_r^2 \lambda_t + \left(\frac{\partial \eta}{\partial \lambda} - 2 \frac{\partial \xi}{\partial r} \right) \lambda_{rr} \\ &\quad - 2 \frac{\partial \tau}{\partial r} \lambda_{rr} - 3 \frac{\partial \xi}{\partial \lambda} \lambda_r \lambda_{rr} - \frac{\partial \tau}{\partial \lambda} \lambda_t \lambda_{rr} \\ &\quad - 2 \frac{\partial \tau}{\partial \lambda} \lambda_r \lambda_{rr}. \end{aligned} \quad (23)$$

Invariance of eq. (12) under eq. (20) gives:

$$\begin{aligned} & (D_*\lambda_{rr} - \lambda_t) + \varepsilon(D_*[\tilde{\eta}_{rr}] - [\tilde{\eta}_t]) \\ & + O(\varepsilon^2) = 0 \end{aligned} \quad (24)$$

Then, substituting for $[\tilde{\eta}_{rr}]$ and $[\tilde{\eta}_t]$ from eqs. (23) and (22), respectively, and substituting for λ_{rr} from eq. (12) gives:

$$\begin{aligned} & D_* \left[\eta_{rr} + (2\eta_{r\lambda} - \xi_{rr})\lambda_r - \tau_{rr}\lambda_t \right. \\ & + (\eta_{\lambda\lambda} - 2\xi_{r\lambda})\lambda_r^2 - 2\tau_{r\lambda}\lambda_r\lambda_t \\ & - \xi_{\lambda\lambda}\lambda_r^3 - \tau_{\lambda\lambda}\lambda_r^2\lambda_t + (\eta_{\lambda} - 2\xi_r) \left(\frac{1}{D_*}\lambda_t \right) \\ & - 2\tau_r\lambda_{rt} - 3\xi_{\lambda}\lambda_r \left(\frac{1}{D_*}\lambda_t \right) - \tau_{\lambda}\lambda_t \left(\frac{1}{D_*}\lambda_t \right) \\ & \left. - 2\tau_{\lambda}\lambda_r\lambda_{rt} \right] - \left[\eta_t + (\eta_{\lambda} - \tau_t)\lambda_t - \xi_t\lambda_r \right. \\ & \left. - \tau_{\lambda}\lambda_t^2 - \xi_{\lambda}\lambda_r\lambda_t \right] = 0 \end{aligned} \quad (25)$$

The solution of eq. (25) gives the infinitesimal elements (ξ, τ, η) leaving invariant eq. (12). As a comparatively simple solution, the following relations were found:

$$\left. \begin{aligned} \xi &= \frac{1}{2}c_3 r + c_5 t + c_6, \\ \tau &= c_3 t + c_4, \text{ and} \\ \eta &= c_2 \lambda - \frac{c_5}{2D_*} r \lambda + c_1 \end{aligned} \right\} \quad (26)$$

where

$c_1, c_2, c_3, c_4, c_5,$ and $c_6 =$ arbitrary constants.

The similarity variables are obtained by solving the characteristics equation (Bluman and Cole, 1974):

$$\frac{dr}{\xi} = \frac{dt}{\tau} = \frac{d\lambda}{\eta} \quad (27)$$

The general solution of eq. (27) involves two constants, one of them becomes the similarity variable and the other plays the role of a new dependent variable. From the integrals of the two equations $dr/\xi = dt/\tau$ and $dt/\tau = d\lambda/\eta$, with $c_3 \neq 0$, the similarity variables are obtained:

$$\left. \begin{aligned} \rho_* &= \frac{1}{\sqrt{4D_*}} \rho_1 = \frac{1}{\sqrt{4D_*}} \frac{r}{\sqrt{t}} \\ &= \frac{r}{2\sqrt{D_*t}} \\ \lambda &= H(\rho_*) \end{aligned} \right\} \quad (28)$$

when substitute in eq. (28) into eq. (12), and rearranging, the result would be:

$$\frac{d^2H}{d\rho_*^2} + 2\rho_* \frac{dH}{d\rho_*} = 0 \quad (29)$$

subjected to:

$$\left. \begin{aligned} \rho_* &= \infty, \quad H(\infty) = 0, \text{ and} \\ \rho_* &= 0, \quad H(0) = r\mu_i \end{aligned} \right\} \quad (30)$$

because $\lambda = H(\rho_*)$ from eq. (28). After separating variables, integrating, and rearranging, eq. (19) is obtained.

RESULTS AND DISCUSSION

In order to utilize the developed exact solution of the soil-water flow equation to predict the radius of a hemispherical wetted soil volume, i.e., radius of wetting front, the solution needs to be verified by comparing the computed values of wetting front radii with available measured experimental values. The numerical solution has also been used for verification. One set of pertinent data has been found in the literature (data gathered by Clothier and Scotter (1982)) and used for verification. These data can be summarized as follows: initial soil water content, $\theta_i = 0.08$, saturated soil water content, $\theta_s = 0.36$, sorptivity, $S = 1.65 \text{ mm} / \text{s}^{1/2}$, the value of $\beta = 8$ which yields $\gamma = 1.44 * 10^{-3}$, and discharge of the emitter, $Q_e = 1.0 * 10^{-7} \text{ m}^3/\text{s}$. **Fig. 2** shows the



locations of the wetting front for a fine sandy loam soil (medium textured) predicted by the developed model and similar values measured experimentally by Clothier and Scotter (1982); values predicted by a numerical model, finite difference method (FDM), developed by Chung (1987); the hemispherical model developed by Ben-Asher, et al., (1986); and the analytical model developed by Clothier and Scotter (1982). The predicted wetting front patterns by the developed model were concentric hemispheric due to neglecting the gravity effect in the absorption solution. For the first 165 minutes, the predicted values of the wetting front patterns agree well with the experimental values measured by Clothier and Scotter (1982). The maximum relative error was (3.6 %). But, after 360 minutes the predicted wetting front patterns deviated from experimentally measured values, the relative error was about (7.9 %). This over prediction by the developed model is due to errors introduced in the solution when neglecting the gravity-force term in Richards' equation. On the other hand, the predicted wetting front patterns moved slightly slower than those predicted by the numerical finite difference model developed by Chung (1987). The maximum relative error between the two sets of values ranged from (0.44 %) at the first minute to (3.9 %) after 360 minutes. This is mainly due to the different basic assumptions adopted in the two models. In addition, the predicted wetting front patterns agrees with both values predicted by the hemispherical model developed by Ben-Asher, et al., (1986) and the analytical model developed by Clothier and Scotter (1982), respectively. Comparing the wetting front patterns predicted by the developed model with the results of both the hemispherical and analytical models gave an average relative error of (0.8 %) and (0.63 %), respectively, for the first 96 minutes. But after 6 hours, this relative error reached (4.5 %) for both models. Therefore, it can be concluded that the developed model generally provided an accurate-enough means to predict the locations of the wetted soil volume from a point source for medium and fine-textured soils. **Fig. 3** shows the effect of initial soil water content on the wetting front location by using the developed model. It is clear from the results shown in the figure that as initial soil water content increases the volume of wetted soil increases when the time is held constant. **Fig. 4** shows the effect of increasing the saturated soil water content on the movement of wetting front by using the developed model. It can be seen from

the results that as the saturated soil water content increases the rate of advance of the wetting front decreases.

CONCLUSIONS

The simulation of water in homogeneous, unsaturated soils is typically accomplished by solving the unsaturated flow equation. The governing unsaturated flow equation was derived by Richards in 1931. The governing unsaturated flow equation is a nonlinear partial differential equation and difficult to solve exactly in closed form. The aim of this research was to solve Richards' equation and analyze water flow from a point source through medium and fine-texture soils. A mathematical procedure has been developed to solve the unsaturated flow equation by applying Kirchhoff's transformation to linearize the equation. It is a transformation of the dependent variable and is a classical tool to solve nonlinear partial differential equations of water flow in unsaturated soil from a point source. The equation has been solved by two methods one solution utilized the similarity techniques and the other utilized Boltzmann's transformation. An exponential function of soil water diffusivity is selected to demonstrate the use of general similarity theory for describing infiltration of water in soil from a point source. The similarity solution of water infiltration for a functional form of soil water diffusivity compares well with the corresponding experimental data and numerical solutions. The results obtained from the analytical solution of Richards' equation were checked with data previously gathered which relate the distance from the point source to boundary of the saturated wetting front. A sensitivity analysis was also conducted to study the effects of major soil parameters on the movement of the saturated wetting front from a point source. The present analytical solution provides reasonable predictions for absorption problems and can be easily extended to general soil-water flow studies.

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$K(\theta)$: unsaturated soil hydraulic conductivity which is a function of volumetric soil water content, L/T.

r : radial distance from the source, L.

t : time, T,

z : gravitational potential head expressed as the depth below soil surface (positive downward), L.

ε : group parameter.

$O(\varepsilon^2)$: order of the parameter, ε .

θ_s : saturated soil water content, L^3/L^3 .

θ_i : initial soil water content, L^3/L^3 ,

∇ : the divergence operator.

$\psi(\theta)$: total water head which is a function of the volumetric soil water content, L.

ρ : the similarity variable.

θ : volumetric soil water content which is a function of location and time, L^3/L^3 .

∇ : the del operator (gradient operator).

ϕ : matric flux potential, L^2/T .

SYMBOLS AND ABBREVIATIONS

$D(\theta)$: unsaturated soil water diffusivity, L^2/T .

D^* : soil water diffusivity of linearized model, L^2/T .

h : matric potential head, L.

Table1: Analytical solution to water flow equation for some cases of soil water diffusivity.

Soil Water Diffusivity	Solution
$D(\theta) = \text{constant}$	$\frac{\theta - \theta_i}{\theta_s - \theta_i} = \text{erfc}(\rho)$
$D(\theta) = \text{variable}$	$\frac{\int_{\theta_i}^{\theta} D(\theta) d\theta}{\int_{\theta_i}^{\theta_s} D(\theta) d\theta} = \text{erfc}(\rho)$
$D(\theta) = D_0 (n+1) \theta^n$	$\frac{\theta^{n+1} - \theta_i^{n+1}}{\theta_s^{n+1} - \theta_i^{n+1}} = \text{erfc}(\rho)$
$D(\theta) = ae^{b\theta}$	$\frac{e^{b\theta} - e^{b\theta_i}}{e^{b\theta_s} - e^{b\theta_i}} = \text{erfc}(\rho)$

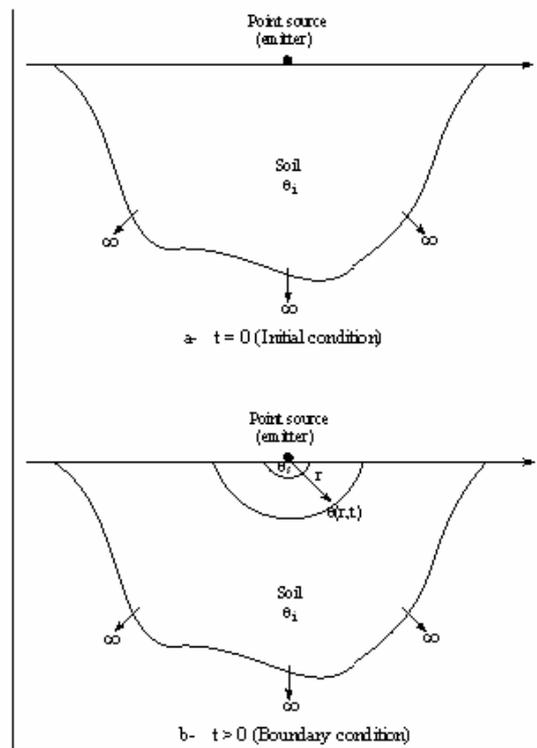


Fig. 1. Physical model of soil water flow.

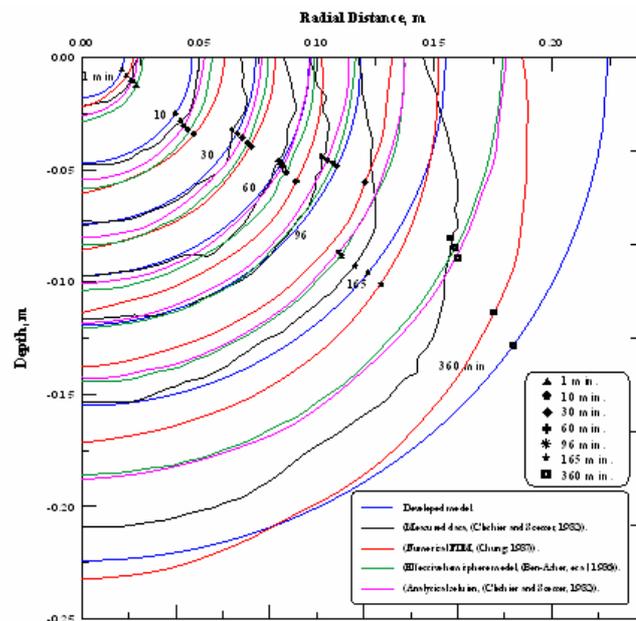


Fig. 2. Wetting front patterns.

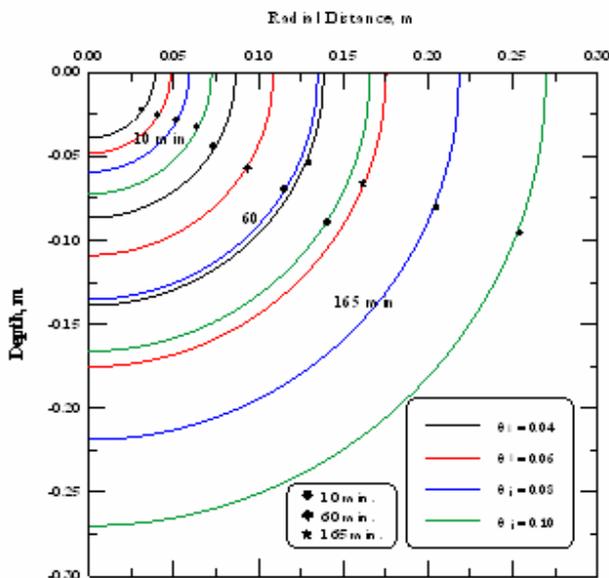


Fig.3. Effect of initial soil water content on the wetting front patterns.

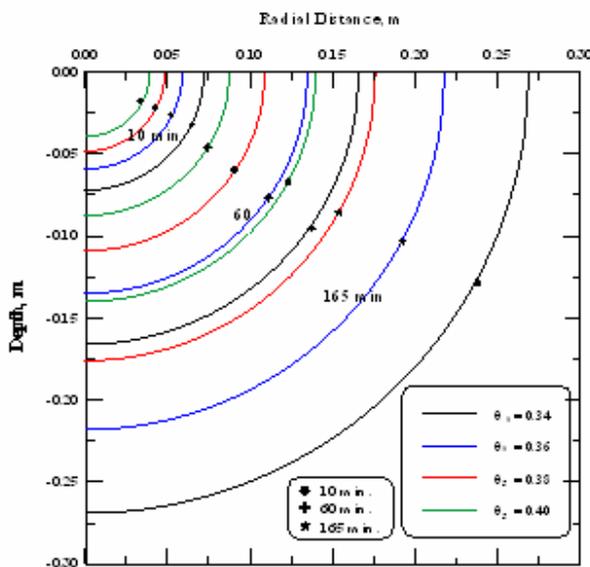


Fig.4. Effect of saturated soil water content on the wetting front patterns.