Exact Stiffness Matrix for Nonprismatic Beams with Parabolic Varying Depth

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ABSTRACT

In this paper, an exact stiffness matrix and fixed-end load vector for nonprismatic beams having parabolic varying depth are derived. The principle of strain energy is used in the derivation of the stiffness matrix. The effect of both shear deformation and the coupling between axial force and the bending moment are considered in the derivation of stiffness matrix. The fixed-end load vector for elements under uniformly distributed or concentrated loads is also derived. The correctness of the derived matrices is verified by numerical examples. It is found that the coupling effect between axial force and bending moment is significant for elements having axial end restraint. It was found that the decrease in bending moment was in the range of 31.72%-42.29% in case of including the effect of axial force for the studied case. For midspan deflection, the decrease was 46.07% due to the effect of axial force generated at supports as a result of axial restraint.

KEYWORDS: Stiffness; Parabolic; Shear deformation; Axial force; Beams

الخلاصة

في هذا البحث، تم اشتقاق مصفوفة الجساءة و متجه حمل النهايات المثبتة للأعتاب الغير موشورية ذات العمق المتغير لاخطيا (قطع مكافىء). تم استعمال مبدأ طاقة الإنفعال في اشتقاق مصفوفة الجساءة مع الأخذ بنظر الإعتبار تأثير تشوهات القص والتأثير المتبادل بين القوة المحورية وعزم الإنحناء. كذلك، تم اشتقاق متجه حمل النهايات المثبتة لعناصر ذات احمال موزعة بإنتظام او أحمال مركزة. تم اختيار عدة امثلة تطبيقية لغرض اثبات صحة المصفوفات التي تم اشتقاقها. من خلال النتائج التي تم الحصول عليها وجد بأن التأثير المتبادل للقوة المحورية مع عزم الإنحناء ذات فاعلية بالنسبة للأعضاء ذات النهايات المقيدة محوريا. فقد لوحظ تناقصا في مقدار عزم الإنحناء وبنسب تتراوح من 31.72% الى 42.29% في حال تم ادخال تأثير القوة المحورية بالنسبة للحالة التي تم دراستها. أما بالنسبة للهطول، فقد كان التناقص بنسبة 46.07% في منتصف الفضاء نتيجة لتأثير القوة المحورية المتولدة من تقييد الحركة المحورية للمساند مقارنة مع الحالة التي تم فيها اهمال تأثيرها.

الكلمات الرئيسية: الجساءة، قطع مكافىء،تشوهات القص، القوة المحورية، الأعتاب

INTRODUCTION

Members with variable depth are used in many engineering structures such as highway bridges, buildings, as well as in many mechanical components and aerospace engineering structures. In civil engineering construction, nonprismatic members are frequently used to optimize material distribution and stresses, increase the overall stability and stiffness, reduce the dead load positive moment and deflection, and sometimes to satisfy architectural requirements. Accordingly, the analysis of structures having nonprismatic elements is of interest in structural, mechanical, and aerospace engineering. The analysis of nonprismatic members is covered in several publications (e.g., Timoshenko and Young 1965; AL-Gahtani 1996; Al-Gahtani and Khan 1998; Luo et al 2007). Most of the available publications deal with the analysis of tapered members only. Some particular cases (e.g., Timoshenko and Young 1965; AL-Gahtani 1996) deal with the analysis of nonprismatic beams having parabolic varying depth. However, these cases are limited to the analysis only (no stiffness matrix derivation) of such type of members involving lengthy and tedious calculations which are not applicable for use in the analysis packages in which the analysis is based on matrix operations. In addition, the available analytical solutions do not consider the effect of shear deformation and the axial force-bending moment interaction. The other alternative publications deal with the numerical methods of analysis such as the finite element method (e.g., Bathe 1996) in which the member is discretized to a number of elements and the stiffness matrices of the elements are assembled to obtain the stiffness matrix for the whole member. The main disadvantage resulting from member discretization is the large number of input data required even for simple structures.

The purpose of this paper is to present an exact stiffness matrix for nonprismatic beam elements with parabolic varying depth including the effect of shear deformation and the axial force-bending moment interaction. The correctness of the derived stiffness matrix and the fixed-end load vector is examined through numerical examples.

PROBLEM STATEMENT

Consider a nonprismatic Euler- Bernoulli beam element of length ***L*** as shown in Fig. 1(a). The element is rectangular in cross-section and has a parabolic varying depth and constant width. Three degrees of freedom are assumed at each node. Only the deformation in the plane of the element and the bending moment about the centroidal main axis are considered. The positive direction of displacements and forces are as shown in Fig. 1(b).

The stiffness components corresponding to the degrees of freedom shown in Fig. 1(b) can be obtained by using Castigliano's second theorem (Boresi, A.P. and Schmidt, R.J. 2003), which states that the deflection caused by an external force is equal to the partial derivative of the strain energy (*U)* with respect to that force. The total strain energy (*U)* for the element shown in Fig. 1(a) including the strain energy caused by bending moment, shear and axial forces can be given by

$$U = \frac{1}{2E}\int_0^L \frac{M_x^2}{I_x}dx + \frac{1.2}{2Gb}\int_0^L \frac{Q_x^2}{h_x^2}dx + \frac{1}{2Eb}\int_0^L \frac{P_x^2}{h_x}dx \quad (1)$$

where M_x, Q_x, P_x, I_x, h_x, are the bending moment, shear force, axial force, moment of inertia and the depth of the element at the distance *x* respectively; *b*, *E*, *G*, are the width of the element, Young's and shear modulus of elasticity, respectively. The bending moment M_x, shear force Q_x, and the axial force P_x can be found from equilibrium as follows

$$M_x = \frac{1}{2}P_i h_0 (cx^2) + Q_i x - M_i \quad (2a)$$

$$Q_x = Q_i \quad (2b)$$

$$P_x = P_i \quad (2c)$$

The moment of inertia and the depth of the element cross section at a distance *x* can be given by

$$I_x = \frac{bh_0^3}{12}(1+cx^2)^3 = I_0(1+cx^2)^3 \,;\, h_x = h_0(1+cx^2)$$

(3)

where h_0=the minimum depth of the element (at the origin), $c=(h_1-h_0)/(h_0L^2)$, and h_1= the maximum depth of the element.

Substituting eqs. (2a), (2b), (2c), and (3) into eq. (1) and after integrations, the following exact expression for the strain energy can be given as

$$U=\frac{1}{2EI_0}\Bigg[P_i^2 I_0(a_0)+\frac{P_i^2 h_0^2}{4}(a_1)-P_i M_i h_0(a_2) + M_i^2(a_3)+P_iQ_i h_0(a_4)-M_iQ_i(a_5) + Q_i^2(a_6+2EI_0K_v)\Bigg] \quad (4)$$

where

$$a_0=\frac{\phi_0}{\sqrt{c}A_0} \quad (5a)$$

$$a_1=\frac{1}{\sqrt{c}}\left[\frac{3}{4}\left(\frac{1}{2}\phi_0-\frac{1}{4}\sin(2\phi_0)\right)-\frac{1}{4}\sin^3\phi_0\cos\phi_0\right] \quad (5b)$$

$$a_2=\frac{1}{4\sqrt{c}}\left[\frac{1}{2}\phi_0-\frac{1}{8}\sin(4\phi_0)\right] \quad (5c)$$

$$a_3=\frac{1}{\sqrt{c}}\left[\frac{3}{4}\left(\frac{1}{2}\phi_0+\frac{1}{4}\sin(2\phi_0)\right)+\frac{1}{4}\cos^3\phi_0\sin\phi_0\right] \quad (5d)$$

$$a_4=\frac{1}{2c}\left[\frac{1}{2(1+cL^2)^2}-\frac{1}{(1+cL^2)}+\frac{1}{2}\right] \quad (5e)$$

$$a_5=\frac{1}{2c}\left[1-\frac{1}{(1+cL^2)^2}\right] \quad (5f)$$

$$a_6=\frac{1}{c}(a_2) \quad (5g)$$

and

$$K_v=\frac{1.2\phi_0}{2GA_0\sqrt{c}}\ ,\ \phi_0=\tan^{-1}(\sqrt{c}\,L)\ ,\text{and}\ A_0=bh_0$$

(6)

The partial derivative of the strain energy (U) with respect to P_i, Q_i, and M_i can be given respectively as follows

$$\frac{\partial U}{\partial P_i}=u_i=\frac{1}{2EI_0}\left[P_i\left(2I_0a_0+\frac{h_0^2}{2}(a_1)\right)+ Q_i\left(h_0(a_4)\right)-M_i\left(h_0(a_2)\right)\right] \quad (7)$$

$$\frac{\partial U}{\partial Q_i}=v_i=\frac{1}{2EI_0}\left[P_i\left(h_0(a_4)\right)+Q_i\left(2(a_6)+4EI_0K_v\right) -M_i(a_5)\right] \quad (8)$$

$$\frac{\partial U}{\partial M_i}=\theta_i=\frac{1}{2EI_0}\left[-P_i\left(h_0(a_2)\right)-Q_i\left(a_5\right)+M_i(2a_3)\right] \quad (9)$$

where u_i, v_i, and θ_i are the displacement components in horizontal ,vertical directions, and rotation angle at node (i) respectively.

The stiffness coefficient (k_{ij}) of an element can be defined as the force or moment at node (i) required to induce a unit displacement or rotation at node (j) with all other displacements equal to zero. Therefore, eqs. (7), (8), and (9) will be used to derive the stiffness matrix of the element.

Writing eqs. (7), (8), and (9) in a matrix form yield the following

$$\begin{bmatrix} (2I_0a_0+\frac{h_0^2}{2}(a_1)) & \left(h_0(a_4)\right) & -\left(h_0(a_2)\right) \\ \left(h_0(a_4)\right) & \left(2(a_6)+4EI_0K_v\right) & -(a_5) \\ -\left(h_0(a_2)\right) & -(a_5) & (2a_3) \end{bmatrix}\begin{Bmatrix} P_i \\ Q_i \\ M_i \end{Bmatrix} = 2EI_0\begin{Bmatrix} u_i \\ v_i \\ \theta_i \end{Bmatrix} \quad (10)$$

or

$$[D]\{F\}=2EI_0\{\delta\}\,;\{F\}=2EI_0[D]^{-1}\{\delta\}$$

(11)

where

$$[D]=\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix} = \begin{bmatrix} (2I_0 a_0+\frac{h_0^2}{2}(a_1)) & (h_0(a_4)) & -(h_0(a_2)) \\ (h_0(a_4)) & (2(a_6)+4EI_0K_v) & -(a_5) \\ -(h_0(a_2)) & -(a_5) & (2a_3) \end{bmatrix} \quad (12)$$

$$\{F\}=\begin{Bmatrix} P_i \\ Q_i \\ M_i \end{Bmatrix};\{\delta\}=2EI_0\begin{Bmatrix} u_i \\ v_i \\ \theta_i \end{Bmatrix} \quad (13)$$

AXIAL STIFFNESS

Applying a unit axial displacement at node (i) with all other displacements equal to zero (i.e. put u_i=1, v_i= 0, and $\theta_i=0$ in the displacements vector $\boldsymbol{\delta}$), the stiffness coefficients corresponding to that displacement can be found by solving eq. (11) for the column matrix $\boldsymbol{F}$, hence

$$k_{11}=P_i=\frac{2EI_0}{\lambda}(d_{33}d_{22}-d_{23}^2) \quad (14a)$$

$$k_{21}=Q_i=\frac{-2EI_0}{\lambda}(d_{33}d_{12}-d_{13}d_{23}) \quad (14b)$$

$$k_{31}=M_i=\frac{2EI_0}{\lambda}(d_{23}d_{12}-d_{13}d_{22}) \quad (14c)$$

From equilibrium, the force vector (or stiffness coefficients) at node (j) corresponding to the unit axial displacement at node (i) can be given as

$$k_{41}=P_j=-k_{11} \quad (14d)$$

$$k_{51}=Q_j=-k_{21} \quad (14e)$$

$$k_{61}=M_j=k_{11}(\frac{h_1-h_0}{2})+k_{21}L-k_{31} \quad (14f)$$

where

$$\lambda=d_{11}\left(d_{33}d_{22}-d_{23}^2\right)-d_{12}(d_{33}d_{12}-d_{13}d_{23})+ d_{13}(d_{23}d_{12}-d_{13}d_{22}) \quad (15)$$

FLEXURAL STIFFNESS

Following the same procedure given for the derivation of axial stiffness, the flexural (translational and rotational) stiffness coefficients can be obtained by applying a unit lateral displacement or a unit rotation (with all other displacements equal to zero) to obtain the translational or rotational stiffness coefficients, respectively.

Hence, by substituting u_i=0, v_i= 1, and $\theta_i=0$ in the displacements vector $\boldsymbol{\delta}$, the translational stiffness coefficients can be given as

$$k_{12}=P_i=\frac{-2EI_0}{\lambda}(d_{33}d_{12}-d_{13}d_{23}) \quad (16a)$$

$$k_{22}=Q_i=\frac{2EI_0}{\lambda}(d_{33}d_{11}-d_{13}^2) \quad (16b)$$

$$k_{32}=M_i=\frac{-2EI_0}{\lambda}(d_{23}d_{11}-d_{13}d_{12}) \quad (16c)$$

From equilibrium, the stiffness coefficients or the force vector at node (j) corresponding to the unit lateral displacement at node (i) (i.e. u_i=0, v_i= 1, and $\theta_i=0$) can be given as

$$k_{42}=-k_{12}=P_j \quad (16d)$$

$$k_{52}=Q_j=-k_{22} \quad (16e)$$

$$k_{62}=M_j=k_{12}(\frac{h_1-h_0}{2})+k_{22}L-k_{32} \quad (16f)$$

Similarly, the rotational stiffness coefficients can be obtained by substituting u_i=0, v_i= 0, and

$\theta_i = 1$ in the displacement vector $[\delta]$ and solving eq. (11) as follows

$$k_{13} = P_i = \frac{2EI_0}{\lambda}(d_{23}d_{12} - d_{13}d_{22}) \quad (17a)$$

$$k_{23} = Q_i = \frac{-2EI_0}{\lambda}(d_{23}d_{11} - d_{13}d_{12}) \quad (17b)$$

$$k_{33} = M_i = \frac{2EI_0}{\lambda}(d_{22}d_{11} - d_{12}^2) \quad (17c)$$

and from equilibrium

$$k_{43} = P_j = -k_{13} \quad (17d)$$

$$k_{53} = Q_j = -k_{23} \quad (17e)$$

$$k_{63} = M_j = k_{13}(\frac{h_1 - h_0}{2}) + k_{23}L - k_{33} \quad (17f)$$

Taking advantage of the symmetry characteristic in the stiffness matrix, and from equilibrium requirements, the other coefficients of the 6*6 stiffness matrix can be given as follows

$$k_{14} = \frac{-2EI_0}{\lambda}(d_{33}d_{22} - d_{23}^2) \quad (18a)$$

$$k_{24} = \frac{2EI_0}{\lambda}(d_{33}d_{12} - d_{13}d_{23}) \quad (18b)$$

$$k_{34} = \frac{-2EI_0}{\lambda}(d_{23}d_{12} - d_{13}d_{22}) \quad (18c)$$

$$k_{44} = \frac{2EI_0}{\lambda}(d_{33}d_{22} - d_{23}^2) \quad (18d)$$

$$k_{54} = \frac{-2EI_0}{\lambda}(d_{33}d_{12} - d_{13}d_{23}) \quad (18e)$$

$$k_{64} = k_{14}(\frac{h_1 - h_0}{2}) + k_{24}L - k_{34} \quad (18f)$$

$$k_{15} = \frac{2EI_0}{\lambda}(d_{33}d_{12} - d_{13}d_{23}) \quad (19a)$$

$$k_{25} = \frac{-2EI_0}{\lambda}(d_{33}d_{11} - d_{13}^2) \quad (19b)$$

$$k_{35} = \frac{2EI_0}{\lambda}(d_{23}d_{11} - d_{13}d_{12}) \quad (19c)$$

$$k_{45} = \frac{-2EI_0}{\lambda}(d_{33}d_{12} - d_{13}d_{23}) \quad (19d)$$

$$k_{55} = \frac{2EI_0}{\lambda}(d_{33}d_{11} - d_{13}^2) \quad (19e)$$

$$k_{65} = k_{15}(\frac{h_1 - h_0}{2}) + k_{25}L - k_{35} \quad (19f)$$

$$k_{16} = k_{11}(\frac{h_1 - h_0}{2}) + k_{21}L - k_{31} \quad (20a)$$

$$k_{26} = k_{12}(\frac{h_1 - h_0}{2}) + k_{22}L - k_{32} \quad (20b)$$

$$k_{36} = k_{13}(\frac{h_1 - h_0}{2}) + k_{23}L - k_{33} \quad (20c)$$

$$k_{46} = -k_{11}(\frac{h_1 - h_0}{2}) - k_{21}L + k_{31} \quad (20d)$$

$$k_{56} = -k_{12}(\frac{h_1 - h_0}{2}) - k_{22}L + k_{32} \quad (20e)$$

$$k_{66} = k_{16}(\frac{h_1 - h_0}{2}) + k_{62}L - k_{36} \quad (20f)$$

where λ is given by eq. (15)

Similar results can be obtained for the stiffness coefficients given by eqs. (18a)-(20f) by using the same procedure presented before. Therefore, substituting for M_x, Q_x, and P_x in the strain energy expression (eq. (1)) interms of the nodal force vector at node (*j*) and following the same previous procedure will yield the same expressions given in eqs. (18a)-(20f).

The obtained stiffness coefficients can be written in a matrix form as

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & sym. & & k_{44} & k_{45} & k_{46} \\ & & & & k_{55} & k_{56} \\ & & & & & k_{66} \end{bmatrix} \quad (21)$$

For a beam element having an orientation as shown in Fig. 2, the stiffness coefficients can be obtained by the same previous procedure. The stiffness matrix for this case can be written interms of the obtained coefficients (eqs. (14a)-(20f)) of the above stiffness matrix as follows

$$\left[\bar{K}\right]=\begin{bmatrix}\bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} & \bar{k}_{15} & \bar{k}_{16}\\ & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} & \bar{k}_{25} & \bar{k}_{26}\\ & & \bar{k}_{33} & \bar{k}_{34} & \bar{k}_{35} & \bar{k}_{36}\\ & sym. & & \bar{k}_{44} & \bar{k}_{45} & \bar{k}_{46}\\ & & & & \bar{k}_{55} & \bar{k}_{56}\\ & & & & & \bar{k}_{66}\end{bmatrix} \quad (22)$$

in which

$$\bar{k}_{11}=\frac{2EI_0}{\lambda}(d_{33}d_{22}-d_{23}{}^2) \quad (23a)$$

$$\bar{k}_{12}=\frac{2EI_0}{\lambda}(d_{33}d_{12}-d_{13}d_{23}) \quad (23b)$$

$$\bar{k}_{13}=k_{14}(\frac{h_1-h_0}{2})+k_{24}L-k_{34} \quad (23c)$$

$$\bar{k}_{14}=\frac{-2EI_0}{\lambda}(d_{33}d_{22}-d_{23}{}^2) \quad (23d)$$

$$\bar{k}_{15}=\frac{-2EI_0}{\lambda}(d_{33}d_{12}-d_{13}d_{23}) \quad (23e)$$

$$\bar{k}_{16}=\frac{-2EI_0}{\lambda}(d_{23}d_{12}-d_{13}d_{22}) \quad (23f)$$

$$\bar{k}_{22}=\frac{2EI_0}{\lambda}(d_{33}d_{11}-d_{13}{}^2) \quad (24a)$$

$$\bar{k}_{23}=k_{12}(\frac{h_1-h_0}{2})+k_{22}L-k_{32} \quad (24b)$$

$$\bar{k}_{24}=\frac{-2EI_0}{\lambda}(d_{33}d_{12}-d_{13}d_{23}) \quad (24c)$$

$$\bar{k}_{25}=\frac{-2EI_0}{\lambda}(d_{33}d_{11}-d_{13}{}^2) \quad (24d)$$

$$\bar{k}_{26}=\frac{-2EI_0}{\lambda}(d_{23}d_{11}-d_{13}d_{12}) \quad (24e)$$

$$\bar{k}_{33}=k_{16}(\frac{h_1-h_0}{2})+k_{62}L-k_{36} \quad (24f)$$

$$\bar{k}_{34}=-k_{14}(\frac{h_1-h_0}{2})-k_{24}L+k_{34} \quad (25a)$$

$$\bar{k}_{35}=-k_{12}(\frac{h_1-h_0}{2})-k_{22}L+k_{32} \quad (25b)$$

$$\bar{k}_{36}=k_{13}(\frac{h_1-h_0}{2})+k_{23}L-k_{33} \quad (25c)$$

$$\bar{k}_{44}=\frac{2EI_0}{\lambda}(d_{33}d_{22}-d_{23}{}^2) \quad (25d)$$

$$\bar{k}_{45}=\frac{2EI_0}{\lambda}(d_{33}d_{12}-d_{13}d_{23}) \quad (25e)$$

$$\bar{k}_{46}=\frac{2EI_0}{\lambda}(d_{23}d_{12}-d_{13}d_{22}) \quad (25f)$$

$$\bar{k}_{55}=\frac{2EI_0}{\lambda}(d_{33}d_{11}-d_{13}{}^2) \quad (26a)$$

$$\bar{k}_{56}=\frac{2EI_0}{\lambda}(d_{23}d_{11}-d_{13}d_{12}) \quad (26b)$$

$$\bar{k}_{66}=\frac{2EI_0}{\lambda}(d_{22}d_{11}-d_{12}{}^2) \quad (26c)$$

For elements having no axial force-bending moment coupling such as when the centroidal axis of the element is straight (i.e. the element is symmetric about centroidal axis), the obtained coefficients can be modified by substituting the following values for the[D]matrix coefficients (eq. (12)) such that

$$\left[D\right]=\begin{bmatrix}d_{11} & d_{12} & d_{13}\\ d_{12} & d_{22} & d_{23}\\ d_{13} & d_{23} & d_{33}\end{bmatrix}=\begin{bmatrix}(2I_0\,a_0) & 0 & 0\\ 0 & \left(2(a_6)+4EI_0K_v\right) & -(a_5)\\ 0 & -(a_5) & (2a_3)\end{bmatrix} \quad (27)$$

FIXED-END LOAD VECTOR DUE TO UNIFORM LOAD

Consider a nonprismatic beam element with a parabolic varying depth under a uniform load q as shown in Fig. 3(a). By using the principle of superposition and knowing that the sum of all displacement components in each direction at the fixed end must be zero, the flexibility matrix equation corresponding to node (i) can be written as

$$\begin{bmatrix} \frac{1}{\sqrt{c}}(\beta_1) & \frac{h_0}{8c}(\gamma) & \frac{-h_0}{2\sqrt{c}}(\gamma_1-\beta) \\ \frac{h_0}{8c}(\gamma) & \frac{1}{c^{3/2}}(\alpha) & \frac{-1}{4c}\left(1-\frac{1}{(1+cL^2)^2}\right) \\ \frac{-h_0}{2\sqrt{c}}(\alpha) & \frac{-1}{4c}\left(1-\frac{1}{(1+cL^2)^2}\right) & \frac{1}{\sqrt{c}}(\beta) \end{bmatrix} \begin{Bmatrix} P_{Fi} \\ Q_{Fi} \\ M_{Fi} \end{Bmatrix} = \frac{q}{c^{3/2}} \begin{Bmatrix} \frac{h_0}{4}(\alpha_1) \\ \frac{\sin^4\phi_0}{8\sqrt{c}} \\ \frac{-1}{2}(\alpha) \end{Bmatrix} \quad (28)$$

where

$$\alpha = \frac{1}{8}\phi_0 - \frac{1}{32}\sin(4\phi_0) \quad (29)$$

$$\beta = \frac{1}{4}\cos^3\phi_0 \sin\phi_0 + \frac{3}{4}\left(\frac{1}{2}\phi_0 + \frac{1}{4}\sin(2\phi_0)\right) \quad (30)$$

$$\gamma = \left(1 + \frac{1}{(1+cL^2)^2} - \frac{2}{(1+cL^2)}\right) \quad (31)$$

$$\alpha_1 = \left(\frac{3}{8}\phi_0 - \frac{1}{4}\sin(2\phi_0) + \frac{1}{32}\sin(4\phi_0)\right) \quad (32)$$

$$\beta_1 = \left(\frac{I_0}{A_0} + \frac{h_0^2}{4}\right)\phi_0 + \frac{h_0^2}{4}(\beta - 2\gamma_1) \quad (33)$$

$$\gamma_1 = \frac{1}{2}\left(\phi_0 + \frac{1}{4}\sin(2\phi_0)\right) \quad (34)$$

Writing the flexibility matrix $[F]$ in eq. (28) in the form

$$[F] = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{c}}(\beta_1) & \frac{h_0}{8c}(\gamma) & \frac{-h_0}{2\sqrt{c}}(\gamma_1-\beta) \\ \frac{h_0}{8c}(\gamma) & \frac{1}{c^{3/2}}(\alpha) & \frac{-1}{4c}\left(1-\frac{1}{(1+cL^2)^2}\right) \\ \frac{-h_0}{2\sqrt{c}}(\alpha) & \frac{-1}{4c}\left(1-\frac{1}{(1+cL^2)^2}\right) & \frac{1}{\sqrt{c}}(\beta) \end{bmatrix} \quad (35)$$

and solving eq. (28) for the unknown fixed end reactions P_{Fi} , Q_{Fi} , and M_{Fi} yields

$$P_{Fi} = \frac{q}{\psi}\left(\frac{h_0\alpha_1}{4c^{3/2}}(f_{33}f_{22} - f_{23}f_{32}) - \frac{\sin^4\phi_0}{8c^2}(f_{33}f_{12} - f_{32}f_{13}) - \frac{\alpha}{2c^{3/2}}(f_{23}f_{12} - f_{22}f_{13})\right) \quad (36a)$$

$$Q_{Fi} = \frac{q}{\psi}\left(\frac{-h_0\alpha_1}{4c^{3/2}}(f_{33}f_{21} - f_{31}f_{23}) + \frac{\sin^4\phi_0}{8c^2}(f_{33}f_{11} - f_{31}f_{13}) + \frac{\alpha}{2c^{3/2}}(f_{23}f_{11} - f_{21}f_{13})\right) \quad (36b)$$

$$M_{Fi} = \frac{q}{\psi}\left(\frac{h_0\alpha_1}{4c^{3/2}}(f_{32}f_{21} - f_{31}f_{22}) - \frac{\sin^4\phi_0}{8c^2}(f_{32}f_{11} - f_{31}f_{12}) - \frac{\alpha}{2c^{3/2}}(f_{22}f_{11} - f_{21}f_{12})\right) \quad (36c)$$

where

$$\psi = f_{11}(f_{22}f_{33} - f_{32}f_{23}) - f_{12}(f_{33}f_{21} - f_{31}f_{23}) + f_{13}(f_{32}f_{21} - f_{31}f_{22}) \quad (37)$$

The right hand side of eq. (28) represents the free-end displacement vector at node (i) due to applied load q.

From equilibrium, the fixed-end reactions at node (j) can be given as
$P_{Fj} = -P_{Fi}$, $Q_{Fj} = qL - Q_{Fi}$, and

$$M_{Fj} = P_{Fi}\left(\frac{h_1 - h_0}{2}\right) + Q_{Fi} - M_{Fi} - \frac{qL^2}{2} \tag{38}$$

FIXED-END LOAD VECTOR DUE TO CONCENTRATED LOAD

For a beam element loaded by a concentrated load (P) at an arbitrary location defined by a distance (L_1) from the left support as shown in Fig. 3(b), the fixed-end load vector can be derived by using the same procedure given before. The flexibility matrix equation for this case can be given as

$$\begin{bmatrix} \frac{1}{\sqrt{c}}(\beta_1) & \frac{h_0}{8c}(\gamma) & \frac{-h_0}{2\sqrt{c}}(\gamma_1 - \beta) \\ \frac{h_0}{8c}(\gamma) & \frac{1}{c^{3/2}}(\alpha) & \frac{-1}{4c}\left(1 - \frac{1}{(1+cL^2)^2}\right) \\ \frac{-h_0}{2\sqrt{c}}(\alpha) & \frac{-1}{4c}\left(1 - \frac{1}{(1+cL^2)^2}\right) & \frac{1}{\sqrt{c}}(\beta) \end{bmatrix} \begin{Bmatrix} P_{Fi} \\ Q_{Fi} \\ M_{Fi} \end{Bmatrix} = P \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} \tag{39}$$

where

$$\delta_1 = \frac{-h_0}{8c}\left(2\left(\frac{1}{(1+cL_1^2)} - \frac{1}{(1+cL^2)}\right) + \left(\frac{1}{(1+cL^2)^2} - \frac{1}{(1+cL_1^2)^2}\right) + 4L_1\sqrt{c}\left((\phi_1 + \beta) - (\phi_0 + \bar{\beta})\right)\right) \tag{40a}$$

$$\delta_2 = \frac{1}{4c^{3/2}}\left(\left(\frac{1}{2}(\phi_0 - \phi_1) - \frac{1}{8}(\sin(4\phi_0) - \sin(4\phi_1)\right) + L_1\sqrt{c}\left(\frac{1}{(1+cL^2)^2} - \frac{1}{(1+cL_1^2)^2}\right)\right) \tag{40b}$$

$$\delta_3 = \frac{-1}{4c}\left(4L_1\sqrt{c}(\beta - \bar{\beta}) + \left(\frac{1}{(1+cL_1^2)^2} - \frac{1}{(1+cL^2)^2}\right)\right) \tag{40c}$$

$$\bar{\beta} = \frac{1}{4}\cos^3\phi_1 \sin\phi_1 + \frac{3}{4}\left(\frac{1}{2}\phi_1 + \frac{1}{4}\sin(2\phi_1)\right),$$
$$\phi_1 = \tan^{-1}(\sqrt{c}\,L_1) \tag{41}$$

and all other constants are previously defined.

Solving eq. (39) yields the following expressions for the unknowns fixed-end reactions at node (i) interms of the flexibility matrix coefficients

$$P_{Fi} = \frac{P}{\psi}\left(\delta_1(f_{33}f_{22} - f_{23}f_{32}) - \delta_2(f_{33}f_{12} - f_{32}f_{13}) + \delta_3(f_{23}f_{12} - f_{22}f_{13})\right) \tag{42a}$$

$$Q_{Fi} = \frac{P}{\psi}\left(-\delta_1(f_{33}f_{21} - f_{31}f_{23}) + \delta_2(f_{33}f_{11} - f_{31}f_{13}) - \delta_3(f_{23}f_{11} - f_{21}f_{13})\right) \tag{42b}$$

$$M_{Fi} = \frac{P}{\psi}\left(\delta_1(f_{32}f_{21} - f_{31}f_{22}) - \delta_2(f_{32}f_{11} - f_{31}f_{12}) + \delta_3(f_{22}f_{11} - f_{21}f_{12})\right) \tag{42c}$$

And from equilibrium, the fixed-end reactions at node (j) can be given as

$P_{Fj} = -P_{Fi}$, $Q_{Fj} = P - Q_{Fi}$,and

$$M_{Fj} = P_{Fi}\left(\frac{h_1 - h_0}{2}\right) + Q_{Fi} - M_{Fi} - P(L - L_1) \tag{43}$$

NUMERICAL EXAMPLES

To verify the correctness of the derived matrices, the following examples are considered.

Example 1

Consider the beam shown in Fig. 4 which has a single span of length, L=1units and fixed at both ends. The beam is carrying a uniformly distributed load, q=1. The depth of the beam is h_0 =1units at the left end and increase parabolically to h_1 =2 units at the right end and has a unit width, b=1 units. The beam was analyzed by Khan and Al-Gahtani (1995) by using the boundary integral method (BIM). Using the same dimensionless data adopted by Al-Gahtani and Khan, the beam is reanalyzed by using the derived expressions for the fixed-end load vector. The results are presented in Table 1.

Example 2

Consider a three-span continuous bridge girder having a parabolic varying depth as shown in Fig. 5. The depth of the girder varies from h=2.5 units at both ends and midspan to h=7.5 units at interior supports. This problem has been analyzed by Timoshenko and Young 1965 ; Al-Gahtani and Khan (1998).Using the same dimensionless data, the problem is reanalyzed by using the derived stiffness matrices and the fixed- end load vector. The analysis results (at nodes 1,2,3,4,and 5) are presented in Table 2 together with those obtained by Timoshenko and Young (1965) ; Al-Gahtani and Khan (1998). It can be seen that when the girder is restraint against horizontal (axial) displacement at supports (i.e. all supports are hinges), the results diverge significantly from that obtained by other methods as given in the last column of Table 2. This is due to the coupling effect between the axial force generated from axial restraint and the bending moment which reduces the displacement, rotations, and bending moments.

Example 3

Finally, consider the beam shown in Fig. 6 which has a span of unit length and a depth varies parabolically from h=1.0 units at left end to h=2.0 units at the right end. The beam is supported at the left end on a translational spring with a stiffness constant of K=10 , fixed at the right end and carrying a concentrated load P=1.0 at the left end. The beam is analyzed by using the derived stiffness matrix interms of the given dimensionless data. The analysis results are presented in Table 3 in which the third column show the results when the shear deformation is considered. The results show a significant effect for shear deformation. This is due to the large translational stiffness relative to the rotational stiffness for this beam.

SUMMARY AND CONCLUSIONS

In this paper, an exact stiffness matrix and fixed-end load vector for beams with parabolic varying depth are derived. An exact integrations were carried out to obtain the strain energy equation including bending, shear, and axial strain energies which is used to obtain the exact expressions for the coefficients of the stiffness matrix. The correctness of the derived expressions is examined through numerical examples. It is found that the derived stiffness matrices and the equivalent load vector are efficient for the analysis of structures having members with parabolic varying depth. Furthermore, the derived matrices can be used in the structural analysis softwars as compared to the available analytical solutions. The obtained results show a significant effect for axial force-bending moment coupling in continuous beams with axial restraint.

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h_x = depth of beam at any section x;

I_0, I_x = moment of inertia of beam cross-section;

K_v = coefficient defined in eq. (6);

$[K]$,$[\bar{K}]$ = stiffness matrices;

L = length of beam;

M, P, Q = bending moment, axial force, and shear force;

U = total strain energy;

u, v = displacements in X and Y directions;

$\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ = variables defined in eqs. (29)-(34);

$\delta_1, \delta_2, \delta_3$ = variables defined in eqs. (40a)-(40c);

θ_i, θ_j = rotational angles at nodes i, j;

λ = variable defined in eq. (15);

ϕ_0, ϕ_1 = variables defined by eq. (6) and eq. (41) respectively; and

ψ = variable defined in eq. (37).

NOTATION

The following symbols are used in this paper:

A_0 = minimum cross-sectional area of beam element;

b = width of beam cross-section;

c = depth variation variable;

$[D]$ = matrix defined by eq. (12)

E = Young's modulus;

$[F]$ = flexibility matrix defined by eq. (35);

G = shear modulus;

h_0, h_1 = minimum ,and maximum depth of beam respectively;

Table 1. Fixed-End Actions

Variable	BIM[a]	Exact[b]
P_{F1}	0.0000	0.0088
Q_{F1}	0.4267	0.4232
M_{F1}	0.0568	0.0564
P_{F2}	0.0000	0.0088
Q_{F2}	0.5733	0.5768
M_{F2}	0.1301	0.1287

[a] Khan and Al-Gahtani (1995)
[b] Present Analysis

Table 2. Supports Reactions, Midspan Deflection, and Angles of rotation

Variable	BIM[a]	Slop-Deflection[b]	Exact[c] (Free horizontal disp.)	Exact[c] (Horizontal disp. is restraint)
P_1	0.000	0.000	0.000	-6.489
P_2	0.000	0.000	0.000	98.978
P_4	0.000	0.000	0.000	-128.116
P_5	0.000	0.000	0.000	35.667
Q_1	1.510	1.500	1.337	7.090
Q_2	72.450	72.850	72.644	66.165
Q_4	46.620	46.150	46.768	40.187
Q_5	-12.580	-12.500	-12.732	-5.433
M_2	-593.750	-594.000	-598.393	-408.562
M_3	124.810	138.600	125.941	72.680
M_4	-452.810	-453.000	-458.562	-283.633
$\bar{v}_3$	-	-	-30086.330	-16224.470
$\bar{\theta}_1$	290.250	376.560	387.600	-59.462
$\bar{\theta}_2$	-423.920	-551.880	-560.712	-212.144
$\bar{\theta}_4$	615.500	800.640	809.900	433.410
$\bar{\theta}_5$	-775.540	-1006.200	-1018.890	-502.100

[a] Al-Gahtani and Khan (1998)
[b] Timoshenko and Young (1965)
[c] Present Analysis
$\bar{v}_3 = EI_0 v_3$, $\bar{\theta}_1 = EI_0 \theta_1$, $\bar{\theta}_2 = EI_0 \theta_2$, $\bar{\theta}_4 = EI_0 \theta_4$, $\bar{\theta}_5 = EI_0 \theta_5$

Table 3. Displacements, rotation, and Support reactions

Variable	Exact (no shear deformation)	Exact (with shear deformation)
u_1	-0.01577	-0.00776
v_1	-0.04950	-0.07515
θ_1	0.09460	0.04658
Q_1	0.4950	0.7515
Q_2	0.5050	0.2485
M_2	-0.5050	-0.2485

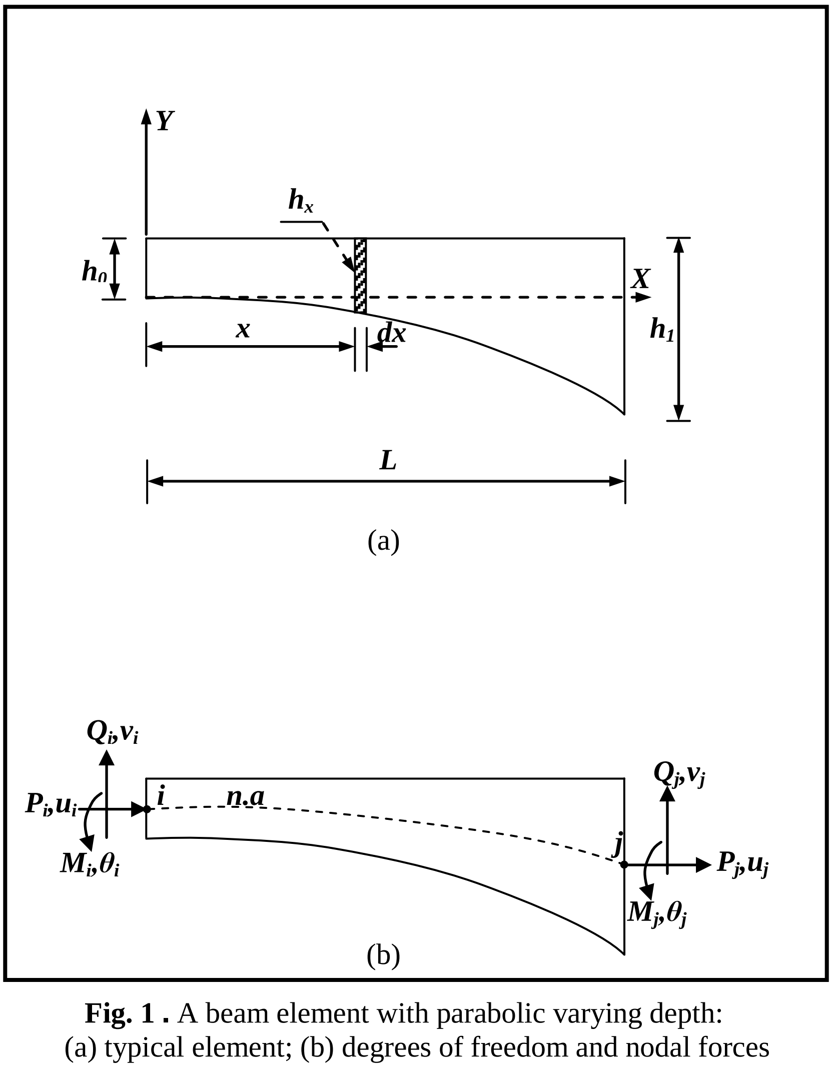

Fig. 1 . A beam element with parabolic varying depth:
(a) typical element; (b) degrees of freedom and nodal forces

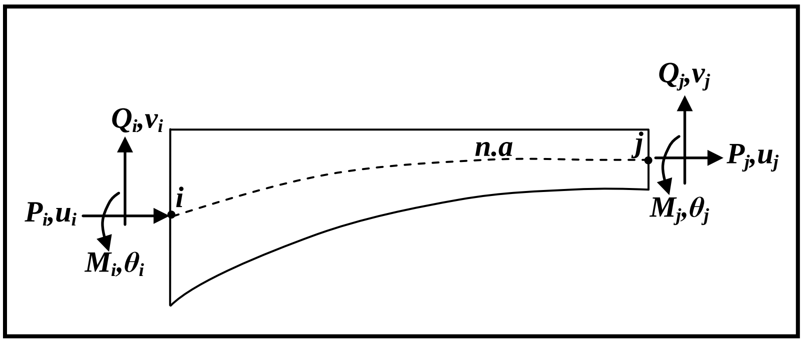

Fig. 2. A beam element with parabolic varying depth

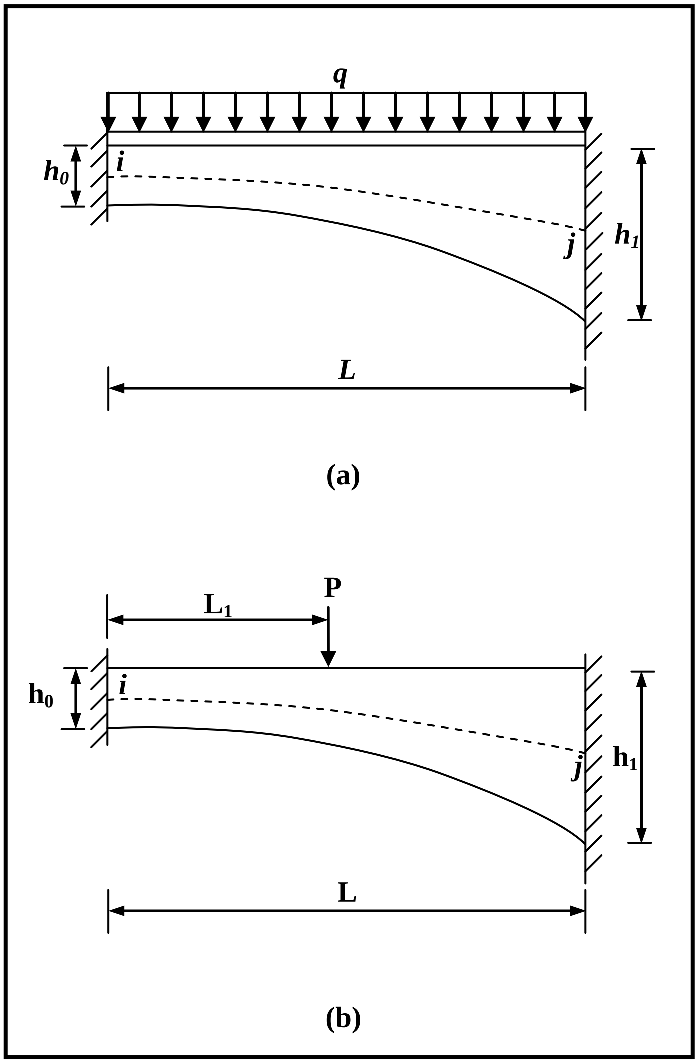

Fig. 3. Beam with parabolic varying depth fixed at both ends under the action of: (a) uniformly distributed load; (b) concentrated load

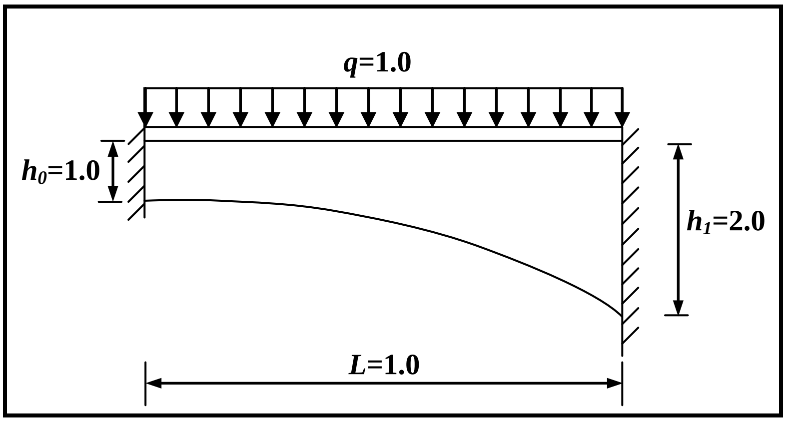

Fig. 4. A Beam with parabolic varying depth (example 1)

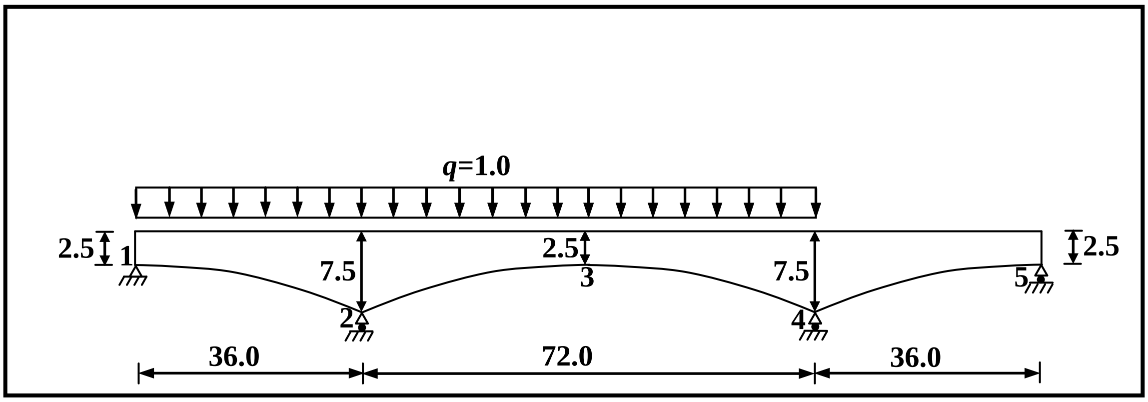

Fig. 5. A three-span continuous bridge girder (example 2)

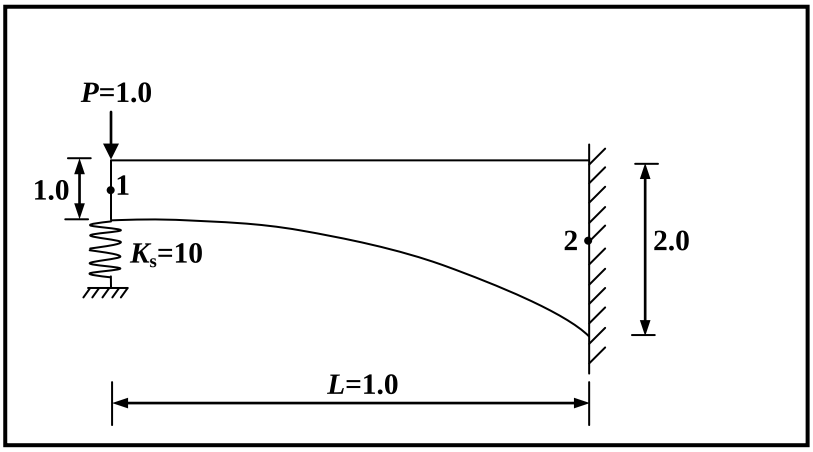

Fig. 6. A Beam with parabolic varying depth elastically supported at one end and fixed at the other end (example 3)