TWO-DIMENSIONAL CONSOLIDATION ANALYSIS OF PARTIALLY SATURATED EARTH EMBANKMENT UNDER ITS SELF WEIGHT BY USING THE FINITE ELEMENT METHOD

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ABSTRACT

This paper studies the two-dimensional analysis of the consolidation process for partially saturated soils applied to an earth embankment mainly from the theoretical point of view. The method of finite elements is used to find a numerical solution to describe the behavior of soils during consolidation. Moreover, this study was carried for both, isothermal and non-isothermal cases.

The case of two–dimensional, plane strain consolidation is considered, as it is widely needed in the studies of soil mechanics. The parameters studied are displacement in the y–direction, pore water pressure, pore air pressure, and temperature.

The embankment was constructed in Basrah - Iraq, where the temperature is usually high at most days of the year. It is believed that it is more realistic to model the embankment as a partially saturated soil rather than a fully saturated soil as was done before.

The study reveals that: –

- The consolidation process is affected by the degree of saturation. As the paths in the soil that are used by the pore–water and/or the pore–air may transfer from one state (open path) to another state (closed path) and vice versa.
- The pore–water and the pore–air pressures are affected by temperature. The pressure increases when the temperature increases (for a constant volume). This is usually the case inside the partially saturated soil. Temperature can affect indirectly the soil skeleton as a result of the change in pore–air and pore–water pressures.
- The vertical displacement for partially saturated soil consolidation under self weight loading is noticed to have almost regular settlement. Then, there is an accelerated settlement due to the dissipation of the air and water pressures. After a while (after excess pressures dissipation), there will be a classical consolidation form.

KEYWORDS

partially saturated soils, unsaturated soils, soil mechanics, geotechnique, consolidation, non-isothermal consolidation

INTRODUCTION

Soils that are unsaturated form the largest category of materials, which do not adhere in behavior to classical, saturated soil mechanics.

The differentiation between saturated and unsaturated soils becomes necessary due to basic differences in their nature and engineering behavior. An unsaturated soil has more than two phases and the pore–water pressure is negative relative to the pore–air pressure.

Any soil near the ground surface, present in a relatively dry environment, will be subjected to negative pore–water pressures and possible desaturation.

It is the presence of more than two phases that results in a material that is difficult to deal with in engineering practice.

An unsaturated soil is commonly defined as having three phases, namely;

- solids,
- water, and
- air.

It may be more correct to recognize the existence of a fourth phase, namely, that of the air–water interface or *contractile skin*. The presence of even the smallest amount of air makes the soil unsaturated. A small amount of air, may occur as occluded air bubbles, which makes the pore fluid compressible. Generally, it is a larger amount of air, which makes the air phase continuous throughout the soil. At the same time, the pore–air and pore–water pressures begin to differ significantly, with the result that the principles and concepts involved differ from those of classical, saturated soil mechanics.

TYPES OF PROBLEMS :–

The types of problems of interest in unsaturated soil mechanics are similar to those of interest in saturated soil mechanics. Common to all unsaturated soil situations are the negative pressures in the pore–water. The type of problem involving negative pore–water pressures that has received the most attention is that of swelling or expansive clays.

Several typical problems are described to illustrate relevant questions, which might be asked by the geotechnical engineer.

- **Construction and Operation of a Dam**
- **Natural Slopes Subjected to Environmental Changes**
- **Mounding Below Waste Retention Ponds**
- **Stability of Vertical or Near Vertical Excavations**
- **Lateral Earth Pressures**
- **Bearing Capacity for Shallow Foundations**
- **Ground Movements Involving Expansive Soils**
- **Collapsing Soils**

CONSTITUTIVE RELATIONS : –

Constitutive relations for an unsaturated soil can be formulated by linking selected deformation state variables to appropriate stress state variables. The deformation state variables must satisfy the continuity requirement. The link of the deformation and stress state variables results in the incorporation of volumetric deformation coefficients. The established constitutive relations can be used to predict volume changes due to changes in the stress state.

Two approaches can be used in establishing the stress versus deformation relationships. These are: the “mathematical” approach and the “semi–empirical” approach. In the mathematical approach, each component of the deformation state variable tensor is expressed as a linear combination of the stress state variables or vice versa. In other words, the relationship between the stress and deformation state variables is expressed by a series of linear equations. The problem with this approach is that it involves the assessment of a large number of soil properties.

The semi–empirical approach involves several assumptions, which are based on experimental evidence from observing the behavior of many materials. These assumptions are that:

- normal stress does not produce shear strain;
- shear stress does not cause normal strain; and
- shear stress component, τ, causes only one shear strain component, (γ).

In addition, the principle of superposition is assumed to be applicable to cases involving small deformations.

The constitutive relations for an unsaturated soil can be formulated as an extension of the equations used for saturated soil, using the appropriate stress state variables. Assume that the soil behaves as an isotropic, linear elastic material. The following constitutive relations are expressed in terms of the stress state variables, ($\sigma - u_a$) and ($u_a - u_w$). The formulation is similar in form to that proposed by Biot in 1941. The soil structure constitutive relations associated with the normal strains in the x–, y–, and z–directions are as follows: –

$$\varepsilon_x = \frac{(\sigma_x - u_a)}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z - 2u_a) + \frac{(u_a - u_w)}{H} \tag{1}$$

$$\varepsilon_y = \frac{(\sigma_y - u_a)}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z - 2u_a) + \frac{(u_a - u_w)}{H} \tag{2}$$

$$\varepsilon_z = \frac{(\sigma_z - u_a)}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y - 2u_a) + \frac{(u_a - u_w)}{H} \tag{3}$$

where

H = modulus of elasticity for the soil structure with respect to a change in matric suction, ($u_a - u_w$).

The constitutive equations associated with the shear deformations are: –

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{4}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{5}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \tag{6}$$

where

τ_{xy} = shear stress on the x–plane in the y–direction (i.e., $\tau_{xy} = \tau_{yx}$),

τ_{yz} = shear stress on the y–plane in the z–direction (i.e., $\tau_{yz} = \tau_{zy}$),

τ_{zx} = shear stress on the z–plane in the x–direction (i.e., $\tau_{zx} = \tau_{xz}$), and

G = shear modulus.

The modulus of elasticity, E, in the above equations is defined with respect to a change in the net normal stress, ($\sigma - u_a$).

The above constitutive equations can also be applied to situations where the stress versus strain curves are nonlinear. Figure 1 shows a typical stress versus strain curve. An incremental procedure using the small increments of stress and

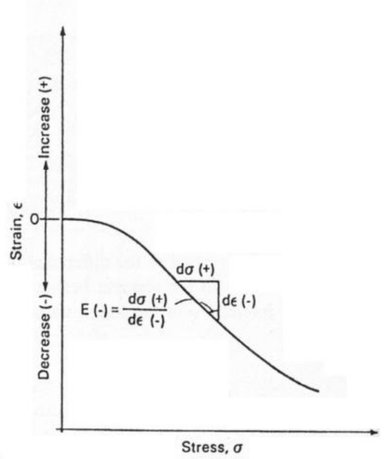


Fig. 1
Nonlinear stress versus strain curve

strain can be used to apply a linear elastic formulation to a nonlinear stress versus strain curve. The nonlinear stress versus strain curve is assumed to be linear within each stress and strain increment. The elastic moduli, E and H, have negative signs, as indicated in figure 1, and may vary in magnitude from one increment to another.

The soil structure constitutive relations associated with normal strains can be written in an incremental form: –

$$d\varepsilon_x = \frac{d(\sigma_x - u_a)}{E} - \frac{\mu}{E} d(\sigma_y + \sigma_z - 2u_a) + \frac{d(u_a - u_w)}{H} \tag{7}$$

$$d\varepsilon_y = \frac{d(\sigma_y - u_a)}{E} - \frac{\mu}{E} d(\sigma_x + \sigma_z - 2u_a) + \frac{d(u_a - u_w)}{H} \tag{8}$$

$$d\varepsilon_z = \frac{d(\sigma_z - u_a)}{E} - \frac{\mu}{E} d(\sigma_x + \sigma_y - 2u_a) + \frac{d(u_a - u_w)}{H} \tag{9}$$

Equations 7, 8, and 9 represent the general elasticity constitutive relations for the soil structure. The left–hand side of the equations refers to a change in the deformation state variable, while its right–hand side contains changes in the stress state variables. A change in the volumetric strain of the soil for each increment, $d\varepsilon_v$, can be obtained by summing the changes in normal strains in the x–, y–, and z–directions:

$$d\varepsilon_v = d\varepsilon_x + d\varepsilon_y + d\varepsilon_z \tag{10}$$

where

$d\varepsilon_v$ = volumetric strain change for each increment.

Substituting equations 7, 8, and 9 into equation 10 gives: –

$$d\varepsilon_v = 3\left(\frac{1-2\mu}{E}\right) d\left(\frac{\sigma_x + \sigma_y + \sigma_z}{3} - u_a\right) + \frac{3}{H} d(u_a - u_w) \tag{11}$$

Equation 11 can be simplified to the following form:–

$$d\varepsilon_v = 3\left(\frac{1-2\mu}{E}\right) d(\sigma_{mean} - u_a) + \frac{3}{H} d(u_a - u_w) \tag{12}$$

where

σ_{mean} = average total normal stress (i.e., $(\sigma_x + \sigma_y + \sigma_z) / 3$).

The volumetric strain change, $d\varepsilon_v$, is equal to the volume change of the soil element divided by the initial volume of the element:

$$d\varepsilon_v = \frac{dV_v}{V_o} \tag{13}$$

The Initial volume, V_o, refers to the volume of the soil element at the start of the volume change process. Therefore, V_o remains constant for all increments. At the end of each increment, the volumetric strain change, $d\varepsilon_v$, can be computed from equation 12 and the volume change of the soil element, dV_v, is obtained from equation 13. The summation of the volumetric strain changes for each increment gives the final volumetric strain of the soil: –

$$\varepsilon_v = \Sigma\, d\varepsilon_v \qquad (14)$$

Change in the Volume of Water

The soil structure constitutive relationship is not sufficient to completely describe the volume changes in an unsaturated soil. Either an air or water phase constitutive relation must be formulated. It is suggested that the water phase is most suitable for formulating the second constitutive relationship. The water phase constitutive relation describes the change in the volume of water present in the referential soil structure element under various stress conditions. The water itself is assumed to be incompressible, and the equation accounts for the net inflow or outflow from the element. The water phase constitutive relation can be formulated in a semi–empirical manner on the basis of a linear combination of the stress state variables. In an incremental form, the constitutive equation can be written as: –

$$\frac{dV_w}{V_o} = \frac{3}{E_w} d\left(\sigma_{mean} - u_a\right) + \frac{d\left(u_a - u_w\right)}{H_w} \qquad (15)$$

where

E_w = water volumetric modulus associated with a change in $(\sigma - u_a)$, and

H_w = water volumetric modulus associated with a change in $(\sigma - u_w)$.

The summation of water volume changes at each increment gives the final change in the volume of water: –

$$\frac{\Delta V_w}{V_o} = \frac{\Sigma dV_w}{V_o} \qquad (16)$$

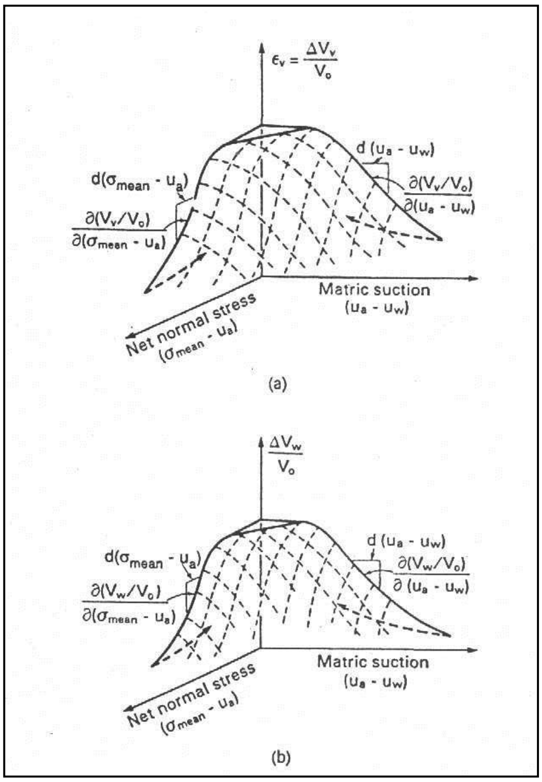
Change in the Volume of Air

The change in the volume of air in an element can be computed as the difference between the soil structure and water volume changes. The continuity requirements (i.e., $\Delta V_v/V_o = \Delta V_w/V_o + \Delta V_a/V_o$) can also be written in an incremental form using the volumetric strain change, $d\varepsilon_v$: –

$$d\varepsilon_v = \frac{dV_w}{V_o} + \frac{dV_a}{V_o} \qquad (17)$$

The constitutive relationships for an unsaturated soil can be presented graphically in the form of constitutive surfaces (figure 2). The deformation state variable is plotted with respect to the $(\sigma_{mean} - u_a)$ and $(u_a – u_w)$ stress state variables. The coefficients used in the constitutive equations are the slopes of the constitutive surface at a point.

The slopes are with respect to both axes. For example, the slopes of the soil structure constitutive surface at a point are equal to $(3\,(1–2\mu)/E)$ and $(3/H)$ with respect to the $(\sigma_{mean} – u_a)$



and $(u_a - u_w)$ axes, respectively. The coefficients on the constitutive surface are referred to as “volumetric deformation coefficients”. These coefficients vary from one stress state to another for a curved constitutive surface. Similarly, the slopes on the water phase constitutive surface at a point are $(3/E_w)$ and $(1/H_w)$ with respect to the $(\sigma_{mean} - u_a)$ and $(u_a - u_w)$ axes, respectively.

The above constitutive relations can be formulated for a general, three–dimensional loading in the x–, y–, and z–directions.

Fig. 2
Three–dimensional constitutive surfaces for an unsaturated soil.
a) soil structure constitutive surface;
b) water phase constitutive surface.
(After Fredlund and Rahardjo, 1993)

Plane Strain Loading

Many geotechnical problems can be simplified into a two–dimensional form using the concept of plane strain or plane stress loading. If an earth structure is significantly long in one direction (e.g., the z–direction) in comparison to the other two directions (e.g., the x– and y–directions) and the loadings are applied only on the x– and y–planes, the structure can be modeled as a plane strain problem. The slope stability, retaining wall, and strip footing are problems commonly analyzed by assuming plane strain loading conditions. For plane strain conditions, the soil deformation in the z–direction is assumed to be negligible $(d\varepsilon_z=0)$. Imposing a condition of zero strain in the z–direction in equation 9 gives: –

$$d(\sigma_z - u_a) = \mu\, d(\sigma_x + \sigma_y - 2u_a) - (E/H)\, d(u_a - u_w) \qquad (18)$$

Volumetric strain during plane strain loading is obtained by substituting equation (18) into equation (11) : –

$$d\varepsilon_v = \frac{2(1+\mu)(1-2\mu)}{E} d(\sigma_{ave} - u_a) + 2\left(\frac{1+\mu}{H}\right) d(u_a - u_w) \qquad (19)$$

where

σ_{ave} = average total normal stress for two–dimensional loading (i.e., $(\sigma_x + \sigma_y)/2$).

Equation 19 can be used as the soil structure constitutive relation for plane strain loading.

The water phase constitutive equation is obtained from equation 15 by replacing the $d(\sigma_z - u_a)$ term with equation 18: –

$$\frac{dV_w}{V_o} = 2\left(\frac{1+\mu}{E_w}\right) d(\sigma_{ave} - u_a) + \left\{\frac{1}{H} - \frac{(E/H)}{E_w}\right\} d(u_a - u_w) \qquad (20)$$

FIELD EQUATIONS AND EQUILIBRIUM ANALYSIS OF AN UNSATURATED SOIL ELEMENT :–

The state of stress at a point in an unsaturated soil can be analyzed using a cubical element of infinitesimal dimensions. Figure 3 illustrates the total normal and shear stresses that act on the boundaries of the soil element. The gravitational force, ρg, (i.e., soil density, ρ, times gravitational acceleration, g) is a body force. The gravitational force acts through the centroid of the element, but is not shown in figure 3 in order to maintain simplicity.

The equilibrium analysis of a soil element is based upon the conservation of linear momentum. The conservation of linear momentum can be applied to the soil element in figure 3 by summing forces first in the y–direction.

$$\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{y}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\rho g\right)dxdydz=0 \qquad (21)$$

where

τ_{xy} = shear stress on the x–plane in the y–direction,

σ_y = total normal stress on the y–plane,

τ_{zy} = shear stress on the z–plane in the y–direction,

ρ = total density of the soil,

g = gravitational acceleration, and

dx, dy, dz = dimensions of the element in the x–, y–, and z-directions respectively.

Since the soil element does not undergo acceleration, the right hand side of equation (21) becomes zero. Equation (21) is commonly referred to as the equilibrium equation for the y–direction. Similarly, equilibrium equations can be derived for the x–direction: –

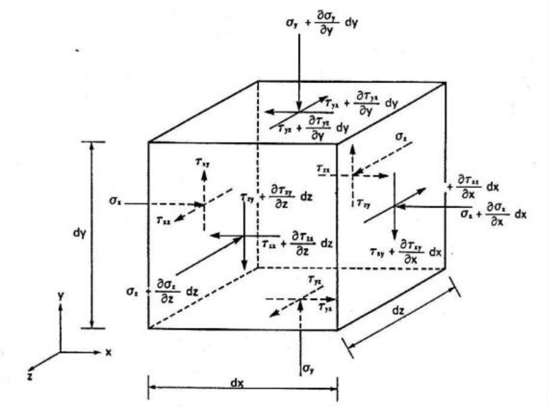


Fig. 3
Normal and shear stresses on a cubical soil element of infinitesimal dimensions.

$$\left(\frac{\partial \sigma_{x}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)dxdydz=0 \qquad (22)$$

Moreover, for the z–direction: –

$$\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{z}}{\partial z}\right)dxdydz=0 \qquad (23)$$

Total or Overall Equilibrium

Total equilibrium refers to the force equilibrium of a complete soil element with its four phases (i.e., air, water, contractile skin, and soil particles). The total stress fields of an unsaturated soil element in the x–, y– and z–directions are presented in figure 4 with the exception that no body forces are shown (i.e., gravitational force, ρg). Only the equilibrium in the y–direction will be analyzed and presented. However, the same principles apply to equilibrium in the x– and z–directions. Figure 4 depicts the total stress fields in the y–direction. The force equilibrium equation associated with figure 4 is given in equation (21).

The water phase equilibrium in the y–direction is obtained by summing forces in the y–direction which gives the equilibrium equation for the water phase.

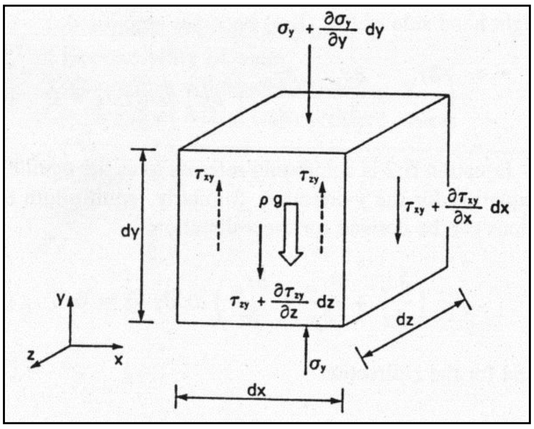


Fig. 4
Components for total equilibrium in the y–direction for an unsaturated soil element

$$\left(n_w \frac{\partial u_w}{\partial y} + n_w \rho_w g + F_{sy}^{w} + F_{cy}^{w} \right) dxdydz = 0 \qquad (24)$$

where

u_w = pore–water pressure,

F^w_{sy} = interaction force (i.e., body force) between the water phase and the soil particles in the y–direction, and

F^w_{cy} = interaction force (i.e., body force) between the water phase and the contractile skin in the y–direction.

The air phase equilibrium in the y–direction is obtained by summing the forces in the y–direction which gives the equilibrium equation for the air phase.

$$\left(n_a \frac{\partial u_a}{\partial y} + n_a \rho_a g + F_{sy}^{a} + F_{cy}^{a} \right) dxdydz = 0 \qquad (25)$$

where

u_a = pore–air pressure,

F^a_{sy} = interaction force (i.e., body force) between the air phase and the soil particles in the y–direction, and

F^a_{cy} = interaction force (i.e., body force) between the air phase and the contractile skin in the y–direction.

Contractile Skin Equilibrium

The contractile skin is only a few molecular layers in thickness. However, its presence affects the equilibrium conditions in an unsaturated soil. This is due to its ability to exert a surface tension, *Ts*.

The equilibrium equation for the contractile skin is as follows:

$$\left\{ -n_c \frac{\partial f^*}{\partial y} (u_a - u_w) - n_c f^* \frac{\partial (u_a - u_w)}{\partial y} + n_c \rho_c g - F_{cy}^{w} - F_{cy}^{a} \right\} dxdydz = 0 \qquad (26)$$

f^* = final interaction between the contractile skin and the soil structure equilibrium

Final Form of Equilibrium Equation

To obtain the final form of the differential equations, some simplifications and assumptions, need to be made, which are: –

1. For two–dimensional analysis, consider the equilibrium of forces that acts in the x–, and y–directions only, and neglect the equilibrium of forces that acts in the z–direction.
2. The value of contractile skin porosity (n_c) is considered very small and approaches to zero in comparison with other values of porosities (i.e., n_s, n_w, and n_a). So $n_c \approx 0$.

The equilibrium equation will be written as: –

$$\frac{\partial(\sigma_x - u_a)}{\partial x} + n_w \frac{\partial(u_a - u_w)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + n_s \frac{\partial u_a}{\partial x} - F_{sx}^{w} - F_{sx}^{a} = 0 \tag{27}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_a)}{\partial y} + n_w \frac{\partial(u_a - u_w)}{\partial y} + n_s \frac{\partial u_a}{\partial y} + n_s \rho_s g - F_{sy}^{w} - F_{sy}^{a} = 0 \tag{28}$$

Mass Transfer

The volume of water entering and leaving the element in y–direction is: –

$$\frac{\partial V_w}{\partial t} = \left(v_w + \frac{\partial v_w}{\partial y} dy \right) dxdz - v_w dxdz \tag{29}$$

where

∂V_w = change in the volume of water in the soil element over a specific time *dt*,

$\partial V_w / \partial t$ = net flux of water through the soil element,

v_w = water flow rate across a unit area of the soil element in the y–direction, and

dx, *dy, dz* = infinitesimal dimensions in the x–, y–, and z–directions, respectively.

The water volume change with respect to time could be written as: –

$$\frac{\partial(V_w/V_o)}{\partial t} = -\frac{k_{wy}}{\rho_w g}\frac{\partial^2 u_w}{\partial y^2} - D_{atm} v \alpha \theta_a \frac{\omega_v}{RT}\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}_v}{\partial y}\right) - \frac{\partial}{\partial y}\left(D_{Tliq}\frac{\partial T}{\partial y}\right)$$
$$-\frac{1}{\rho_w g}\frac{\partial k_{wy}}{\partial y}\frac{\partial u_w}{\partial y} - \frac{\partial k_{wy}}{\partial y} \tag{30}$$

where

k_{wy} = coefficient of permeability with respect to water as a function of matrix suction which varies with location in the y–direction [i.e., $k_{wy}(u_a – u_w)$],

ω_i = molecular mass of the diffusing constituent,

R = universal (molar) gas constant,

T = absolute temperature,

D_{atm} = is the molecular diffusivity of water vapor in air, $cm^2\ sec^{-1}$, D_{atm} could be obtained by applying the equation $D_{atm} = 5.893 \times 10^{-6} T^{2.3} / u_a$ (31)

α = the tortuosity factor allowing for extra path length (~ 0.66),

v = is the mass flow factor introduced to allow for the mass flow of vapor arising from the difference in boundary conditions governing the air and vapor components of the diffusion system.

$$v = \frac{u_a}{(u_a - u_v)} \tag{32}$$

θ_a = volumetric air content which describes the area available for the vapor to flow, and is equal to (1–S) n.

The water phase constitutive relation for plane strain conditions describes the time derivative of the water volume change as: –

$$\frac{d\left(V_w/V_o\right)}{dt}=\left(\frac{1+\mu}{E_w}\right)\frac{d}{dt}\left(\sigma_x+\sigma_y-2u_a\right)+\left\{\frac{1}{H_w}-\frac{\left(E/H\right)}{E_w}\right\}\frac{d}{dt}\left(u_a-u_w\right) \tag{33}$$

Equating with equation (30) gives: –

$$\left(\frac{1+\mu}{E_w}\right)\frac{d}{dt}\left(\sigma_x+\sigma_y-2u_a\right)+\left\{\frac{1}{H_w}-\frac{\left(E/H\right)}{E_w}\right\}\frac{d}{dt}\left(u_a-u_w\right)=-\frac{k_{wy}}{\rho_w g}\frac{\partial^2 u_w}{\partial y^2}-D_{atm}v\alpha\theta_a\frac{\omega_v}{RT}\frac{\partial}{\partial y}\left(\frac{\partial \overline{u}_v}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(D_{Tliq}\frac{\partial T}{\partial y}\right)-\frac{1}{\rho_w g}\frac{\partial k_{wy}}{\partial y}\frac{\partial u_w}{\partial y}-\frac{\partial k_{wy}}{\partial y} \tag{34}$$

For the case of nearly saturated soils, the second term on the right hand side does not exist, because air will be entrapped as bubbles in the water. Therefore, there is no continuous path for the air phase to move from one place to another. Consequently, there is no vapor movement, as a result, there is no contribution of water vapor to the water mass transfer, so, this term can be omitted for nearly saturated soils. For homogeneous soil, the terms containing $\partial k_{wy}/\partial y$ are equal to zero, so the fourth and fifth terms on the right hand side vanish. In addition to the assumptions mentioned, it is assumed that the water density remains constant (incompressible) and the thermal liquid diffusivity also remains constant. The equation of two–dimensions can be written as:–

$$m_1^w\frac{d}{dt}\left(\frac{\sigma_x+\sigma_y}{2}-u_a\right)+m_2^w\frac{d}{dt}\left(u_a-u_w\right)=$$
$$-\frac{1}{\rho_w g}\left[k_{wx}\frac{\partial^2 u_w}{\partial x^2}+k_{wy}\frac{\partial^2 u_w}{\partial y^2}\right]-D_{Tliqx}\frac{\partial^2 T}{\partial x^2}-D_{Tliqy}\frac{\partial^2 T}{\partial y^2} \tag{35}$$

Air Flow Differential Equation

The air phase is compressible and flows in response to an air pressure gradient and a concentration gradient. Therefore, the flow of air through a referential element of unsaturated soil is computed in terms of mass rate of air flow, J_a. The net mass rate of air flow across the element is obtained as the difference between the mass rate of air entering and leaving the element within a period of time: –

$$\frac{\partial M_a}{\partial t}=\left(J_a+\frac{\partial J_a}{\partial y}dy\right)dxdz-J_a dxdz \tag{36}$$

where
$\partial M_a/\partial t$ = net mass rate of air flowing through the soil element, and
J_a = mass rate of air flowing across a unit area of the soil.
The net mass rate of air flow can be expressed for a unit volume of the soil elements as: –

$$\frac{\partial M_a}{\partial t} = \frac{\partial J_a}{\partial y} dydxdz \tag{37}$$

or

$$\frac{\partial (M_a / V_o)}{\partial t} = \frac{\partial J_a}{\partial y} \tag{38}$$

where

$\partial (M_a/V_o)/ \partial t$ = net mass rate of air flow per unit volume of the element, and
V_o = initial total volume of the element (*dxdydz*).

According to Fick's law (Hillel, 1971): –

$$J_a = -D_a \frac{\partial C}{\partial y} \tag{39}$$

where

J_a = mass rate of air flowing across a unit area of the soil,
D_a = transmission constant for air flow through a soil,
C = concentration for the air expressed in terms of the mass of air per unit volume of soil, and
$\partial C/\partial y$ = concentration gradient in the y direction.

The negative sign indicates that air flows in the direction of decreasing concentration gradient.
The change of air volume could be described as: –

$$\frac{\partial (V_a / V_o)}{\partial t} = -\frac{1}{\rho_{ao}} D_a^* \nabla^2 u_a - \frac{1}{\rho_{ao}} \left(\frac{\partial u_a}{\partial x} \frac{\partial D_a^*}{\partial x} + \frac{\partial u_a}{\partial y} \frac{\partial D_a^*}{\partial y} \right) - \frac{(1-S)n}{\bar{u}_{ao}} \frac{\partial \bar{u}_a}{\partial t} + \frac{(1-S)n}{T} \frac{\partial T}{\partial t} \tag{40}$$

From the constitutive relation, the change of air volume could be described as: –

$$\frac{dV_a}{V_o} = \frac{d(\sigma_x - u_a)}{E_a} + \frac{d(\sigma_y - u_a)}{E_a} + \frac{d(\sigma_z - u_a)}{E_a} + \frac{d(u_a - u_w)}{H_a} \tag{41}$$

Simplifying yields: –

$$\frac{d(V_a / V_o)}{dt} = (m_1^s - m_1^w) \frac{d}{dt} \left(\frac{\sigma_x + \sigma_y}{2} - u_a \right) + (m_2^s - m_2^w) \frac{d}{dt} (u_a - u_w) \tag{42}$$

Now, equating and simplifying yields: –

$$(m_1^s - m_1^w) \frac{d}{dt} \left(\frac{\sigma_x + \sigma_y}{2} - u_a \right) + (m_2^s - m_2^w) \frac{d}{dt} (u_a - u_w) = -\frac{1}{\rho_{ao}} D_a^* \nabla^2 u_a$$
$$- \frac{1}{\rho_{ao}} \left(\frac{\partial u_a}{\partial x} \frac{\partial D_a^*}{\partial x} + \frac{\partial u_a}{\partial y} \frac{\partial D_a^*}{\partial y} \right) - \frac{(1-S)n}{\bar{u}_{ao}} \frac{\partial \bar{u}_a}{\partial t} + \frac{(1-S)n}{T} \frac{\partial T}{\partial t} \tag{43}$$

The above derivation is valid when the degree of saturation is below about 90%, where continuous paths for the air phase exist to transfer from one place to another.

For the special case of unsaturated soils where the degree of saturation is about >90%, the air phase exists in the form of occluded bubbles and cannot transfer easily, the air transfer takes place by diffusion.

When the air diffuses in the water (as dissolved air in the water phase due to its pressure), the water phase plays a big role in the transport mechanism for the air phase. When the dissolved air reaches a suitable environment (where the air pressure is less), air is freed from the water phase to its original form. The derivation of the continuity equation for this condition is slightly different.

Air Diffusion Through Water

Fick's law can be used to describe the diffusion process. The concentration gradient, which provides the driving potential for the diffusion process is expressed with respect to the soil voids (i.e., air and water phases). In other words, the mass rate of diffusion and the concentration gradient are expressed with respect to a unit area and a unit volume of the soil voids, respectively.

The formulation of Fick's law for diffusion in the y–direction is as follows: –

$$\frac{\partial M}{\partial t} = -D\frac{\partial C}{\partial y} \tag{44}$$

where

$\partial M/\partial t$ = mass rate of the air diffusing across a unit area of the soil voids,

D = coefficient of diffusion,

C = concentration of the diffusing air expressed in terms of mass per unit volume of the soil voids, and

$\partial C/\partial t$ = concentration gradient in the y–direction (or similarly in the x– or z–directions).

The diffusion equation can appear in several forms, similar to the forms presented for the flow of air through a porous medium. The concentration gradient for gases or water vapor (i.e., $\partial C/\partial y$) can be expressed in terms of their partial pressures. Consider a constituent diffusing through the pore–water in a soil. Equation (44) can be rewritten with respect to the partial pressure of the diffusing constituent: –

$$\frac{\partial M}{\partial t} = -D\frac{\partial C}{\partial \bar{u}_i}\frac{\partial \bar{u}_i}{\partial y} \tag{45}$$

where

$\bar{u}_i$ = partial pressure of the diffusing constituent,

$\partial C/\partial \bar{u}_i$ = change in concentration with respect to a change in partial pressure, and

$\partial \bar{u}_i/\partial y$ = partial pressure gradient in the y–direction (or similarly in the x– or z–directions).

Simplifying for two–dimensions will yield: –

$$\frac{\partial (V_a/V_o)}{\partial t} = -\frac{Dh}{\bar{u}_{ao}}\left(\frac{\partial^2 \bar{u}_a}{\partial x^2} + \frac{\partial^2 \bar{u}_a}{\partial y^2}\right) + \frac{Dh}{T_o}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{46}$$

Equating this expression to the expression in equation 42 yields: –

$$\left(m_1^s - m_1^w\right)\frac{d}{dt}\left(\frac{\sigma_x+\sigma_y}{2} - u_a\right) + \left(m_2^s - m_2^w\right)\frac{d}{dt}\left(u_a - u_w\right) = -\frac{Dh}{\bar{u}_{ao}}\left(\frac{\partial^2 \bar{u}_a}{\partial x^2} + \frac{\partial^2 \bar{u}_a}{\partial y^2}\right) + \frac{Dh}{T_o}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad (47)$$

Heat transfer

The Fourier diffusion equation can be used to describe heat transfer in soils. For one–dimensional flow, the heat flux can be described as: –

$$H = \lambda \frac{\partial T}{\partial y} \quad (48)$$

where

H = heat flux,

λ = thermal conductivity, and

T = temperature.

The heat transfer can be due to conduction, convection, and radiation. In soil, convection and radiation are considered to be negligible (Philip and de Vries, 1957; Lewis and Schrefler, 1987; Edgar et al., 1989; Al–Khaddar, 2000). So the conduction term is considered, in addition to the heat lost due to evaporation of water. This could be described as: –

$$\zeta \frac{\partial T}{\partial t} = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) - L_v \frac{\partial}{\partial y}\left(D_v \frac{\partial u_v}{\partial y}\right) \quad (49)$$

where

ζ = volumetric specific heat of the soil as a function of water content (J/m^3/°C), and

L_v = latent heat of vaporization of water (i.e., 2418000 J/kg).

5. Finite Element Equations :–

Using the method of weighted residuals on equations (27), (28), (35), (47), and (49), a discretized finite element form is reached (Zainal, 2002). The final equations are obtained after simplifying as:–

$$[\mathbf{K}]\frac{\partial \mathbf{u}}{\partial t} + [\mathbf{L}]\frac{\partial \mathbf{u_a}}{\partial t} + [\mathbf{M}]\frac{\partial \mathbf{u_w}}{\partial t} = [\mathbf{f}] \quad ...(50)$$

$$[\mathbf{W1}]\frac{\partial \mathbf{u}}{\partial t} + [\mathbf{W2}]\frac{\partial \mathbf{u_a}}{\partial t} + [\mathbf{W3}]\frac{\partial \mathbf{u_w}}{\partial t} + [\mathbf{KW}]\mathbf{u_w} + [\mathbf{UT}]\mathbf{T} = [\mathbf{FW}] \quad ...(51)$$

$$[\mathbf{A1'}]\frac{\partial \mathbf{u}}{\partial t} + [\mathbf{A2'}]\frac{\partial \mathbf{u_a}}{\partial t} + [\mathbf{A3'}]\frac{\partial \mathbf{u_w}}{\partial t} + [\mathbf{KAUa}]\mathbf{u_a} = \mathbf{FA'} \quad ...(52)$$

$$[\mathbf{T1}]\frac{\partial \mathbf{T}}{\partial t} + [\mathbf{KH}]\mathbf{T} = [\mathbf{FH}] \quad ...(53)$$

or in matrix form, the equations are represented as:–

$$\begin{bmatrix} \mathbf{K} & \mathbf{M} & \mathbf{L} & \mathbf{0} \\ \mathbf{W1} & \mathbf{W3} & \mathbf{W2} & \mathbf{0} \\ \mathbf{A1'} & \mathbf{A3'} & \mathbf{A2'} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T1} \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} \mathbf{u} \\ \mathbf{u_w} \\ \mathbf{u_a} \\ \mathbf{T} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{KW} & \mathbf{0} & \mathbf{UT} \\ \mathbf{0} & \mathbf{0} & \mathbf{KAUa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{KH} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{u_w} \\ \mathbf{u_a} \\ \mathbf{T} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{FW} \\ \mathbf{FA'} \\ \mathbf{FH} \end{Bmatrix} \quad ...(54)$$

After discretization in time domain, the equations are represented as:–

$$\begin{bmatrix} [\mathbf{K}] & [\mathbf{M}] & [\mathbf{L}] & \mathbf{0} \\ [\mathbf{W1}] & [\mathbf{W3}]+\boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t}[\mathbf{KW}] & [\mathbf{W2}] & \boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t}[\mathbf{UT}] \\ [\mathbf{A1'}] & [\mathbf{A3'}] & [\mathbf{A2'}]+\boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t}[\mathbf{KAUa}] & \boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t}[\mathbf{KAT}] \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [\mathbf{T1}]+[\mathbf{KH}]\boldsymbol{\alpha}\boldsymbol{\Delta}\mathbf{t} \end{bmatrix}_{k,\alpha} \begin{Bmatrix} \mathbf{u} \\ \mathbf{u_w} \\ \mathbf{u_a} \\ \mathbf{T} \end{Bmatrix}_{t_k+\Delta t_k} =$$

$$\begin{bmatrix} [\mathbf{K}] & [\mathbf{M}] & [\mathbf{L}] & \mathbf{0} \\ [\mathbf{W1}] & [\mathbf{W3}]-(1-\boldsymbol{\alpha})\boldsymbol{\Delta}\mathbf{t}[\mathbf{KW}] & [\mathbf{W2}] & -(1-\boldsymbol{\alpha})\boldsymbol{\Delta}\mathbf{t}[\mathbf{UT}] \\ [\mathbf{A1'}] & [\mathbf{A3'}] & [\mathbf{A2'}]-(1-\boldsymbol{\alpha})\boldsymbol{\Delta}\mathbf{t}[\mathbf{KAUa}] & -(1-\boldsymbol{\alpha})\boldsymbol{\Delta}\mathbf{t}[\mathbf{KAT}] \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [\mathbf{T1}]-(1-\boldsymbol{\alpha})\boldsymbol{\Delta}\mathbf{t}[\mathbf{KH}] \end{bmatrix}_{k,\alpha} \begin{Bmatrix} \mathbf{u} \\ \mathbf{u_w} \\ \mathbf{u_a} \\ \mathbf{T} \end{Bmatrix}_{t_k}$$

$$+\begin{Bmatrix} [\mathbf{f}] \\ [\mathbf{FW}] \\ [\mathbf{FA'}] \\ [\mathbf{FH}] \end{Bmatrix}\Delta t_k \qquad \text{...(55)}$$

Appendix (A) gives the definitions for the matrices encountered in equations 50 to 55. A finite element program is set to deal with solving equation 55 (Zainal, 2002), the results of which are presented in the next section.

Problem Description

The Mdaina trial embankment was constructed in the southern part of Iraq at Al–Basrah province in 1979. The height of the embankment is 5 meters, the width at the top is 12 meters, and the width at the bottom is 37 meters with side slopes of 2.5:1.

Figure 5 shows the cross section that has been adopted for this analysis. For more details, refer to Penman and Burford (1983).

Figure 6 shows the finite element discretization using quadratic quadrilateral of the serendipity type element. This figure shows the right half of the embankment where the height is 5 meters, the top width is 6 meters, and the bottom width is 18.5 meters.

Due to symmetry, the analysis of the left half is assumed to be identical to the analysis of the right half. A one way drainage condition is assumed because of the clayey soil underneath the embankment and the open surface from above.

Material properties shown in table 1 are after Al–Hamrany (1980) and Appendix (B). The thermal degrees are taken depending on the thermal map of Iraq for dry and wet seasons. Temperature is taken for the dry season as 35.8 °C for the upper part of the embankment, and 32 °C for the lower part of the embankment (Zainal, 2002).

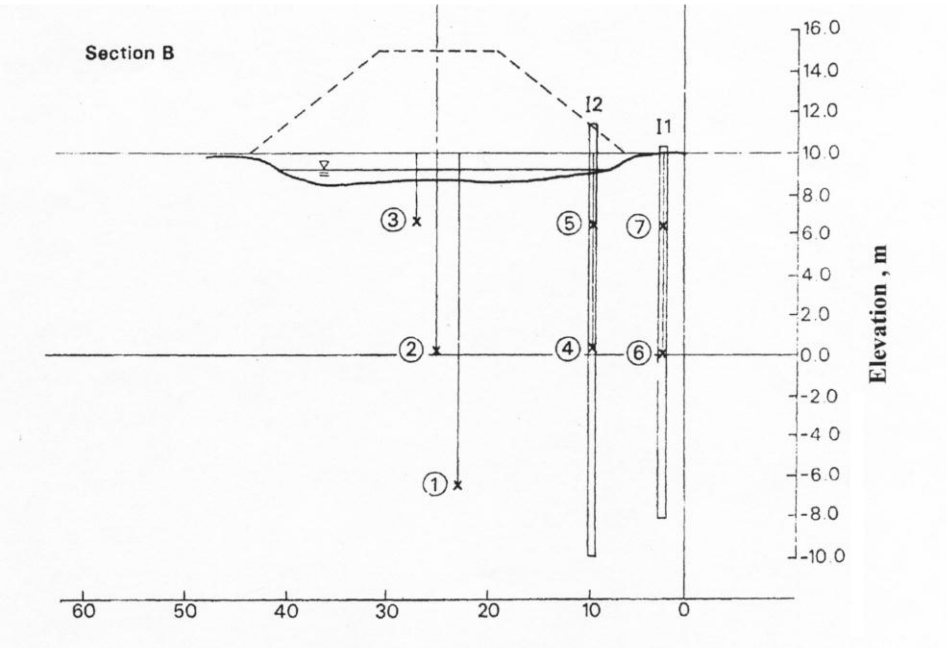


Fig 5
Cross-section of the Mdaina trial embankment

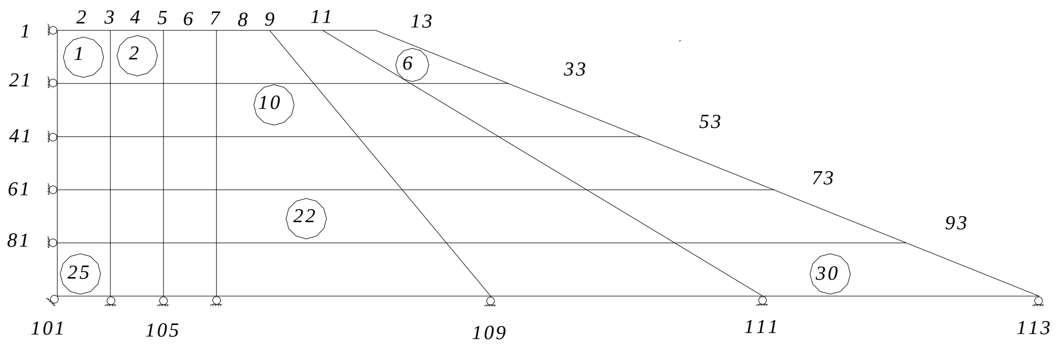


Fig6
Discretized cross section for the finite element method for half of the Mdaina trial embankment

Soil temperature under 1.0 m depth is considered to be the same as for the 1.0 m depth without a significant loss of accuracy (Al–Khaddar, 2000).

The degree of saturation was assumed 0.95 under the assumption that for clayey soils (like the kind of soils used for this embankment) water may reach a large height due to capillarity. Hence, the soil becomes nearly saturated.

Table 1

Material Properties for the Mdaina Consolidation Problem

Property	**Value**	**Units**
Modulus of elasticity E	2870	kN/m^2
Poisson's ratio	0.4	–
Solubility of air in water	0.01708	–
Porosity	50 %	–
Thermal conductivity, λ	72.3168	kJ/°C m day
Degree of saturation, S	90 %	–
Molecular mass of air, ω_a	28.966	kg/kmol
Hydraulic conductivity, k_w	5.76×10^{-8}	cm/sec
Universal gas constant, R	8.31432	J/mol °K
Coefficient of Diffusion, D	1.728×10^{-4}	m^2/day
Volumetric specific heat capacity of the soil solids, ζ_s	2.235×10^6	J/m^3/°C
Volumetric specific heat capacity of the soil solids, ζ_w	4.154×10^6	J/m^3/°C
Temperature coefficient of surface tension of water, γ	-2.09×10^{-3}	–

This section is intended to demonstrate the application of the coupled differential equations that were derived previously for two–dimensional consolidation of a saturated soil.

An isoparametric quadrilateral element of the serendipity type is used in solving this problem by the finite element method. An eight–nodded element is used for all degrees of freedom, which are: –

1. displacement in the y–direction,
2. pore water pressure u_w,
3. pore air pressure u_a, and
4. temperature.

A special computer program was designed and implemented to solve this problem. A FORTRAN 90 compiler was used (Power Station) due to its ease when dealing with large matrices and equation solving facilities.

Results

Results of the non–isothermal consolidation of the trial embankment under its self weight are shown in figures 7 to 11.

Figure 7 shows the temperature variation for various nodes at the centerline of the embankment. Figures 8a, b, and c show the temperature distribution inside the embankment after 1, 10, and 100 days, respectively.

Figures 9a, b, and c show the variation of the pore–water pressure at different time factors (for the water phase) $T_{vw} = 0.1$, $T_{vw} = 0.5$, and $T_{vw} = 0.9$.

Figures 10a, b, and c show the variation of the pore–air pressure at different time factors (for the air phase) $T_{va} = 0.1$, $T_{va} = 0.5$, and $T_{va} = 0.9$.

Figure 11 shows the vertical displacement of various nodes at the centerline of the embankment.

The figures mentioned can be used to describe an elementary analysis of the consolidation behavior for the partially saturated embankment under its own self-weight.

Discussion

Figure 7 shows the temperature variation for various nodes at the centerline of the embankment. The difference between the surface of the embankment and its base is already small (less than 4 °C). After about 500 days, the difference is less than 1 °C.

This means that the increase in the water and air pressures inside the embankment due to the increase in temperature can be assumed negligible after 500 days (for this case study). After that, water and air pressure dissipations are continued normally.

Figures 8a, b, and c show the gradual increase of the temperature inside the embankment. The contour lines in figure 8a shows little roughness due to variations in the assumed initial degrees of temperature at some nodes.

Figures 9a, b, and c show the water–pressure inside the embankment in terms of u_w/u_{wmax} (water pressure / maximum water pressure at 0.1 T_{vw}). The dissipation seems to be going smoothly with time (at different time factors for the water phase) as expected for the fully saturated soil.

Figures 10a, b, and c show the air–pressure inside the embankment in terms of u_a/u_{amax} (air pressure / maximum air pressure at 0.1 T_{va}).

The dissipation of the air pressure also seems to be going smoothly with time (at different time factors for the air phase).

At the early stages of consolidation, there is no obvious pattern of sudden increase in the pore–air pressure at specific places (as previously seen in the one–dimensional and two–dimensional problems that were analyzed earlier) (Al-Damluji et al., 2003).

This could be due to the lack of sufficient load that is considered to be the main cause of the increase in the pore–air pressure near the places of applied load. The only load used for this analysis is the self–weight.

Figure 11 shows the vertical displacement for some specified nodes at the centerline of the embankment. The displacement is represented as y/y_{max} (the displacement of the node / maximum displacement of the surface node).

It is noticed from this figure that the difference in the displacement is not constant even when the vertical distances between the nodes under consideration (nodes 1, 21, 41, 61, and 81 in figure 6) are constant (1 m).

This is due to the difference in the vertical load (self–weight) over a specified node (e.g., load over node 21 due to self–weight is much less than the load over node 81 due to self–weight). Obviously, this will make (for example) the difference in settlement between nodes 1 and 21 much less than the difference in settlement between nodes 61 and 81 (as seen in figure 11).

There is still a little initial compression in the air phase as seen by the values of the settlement at the

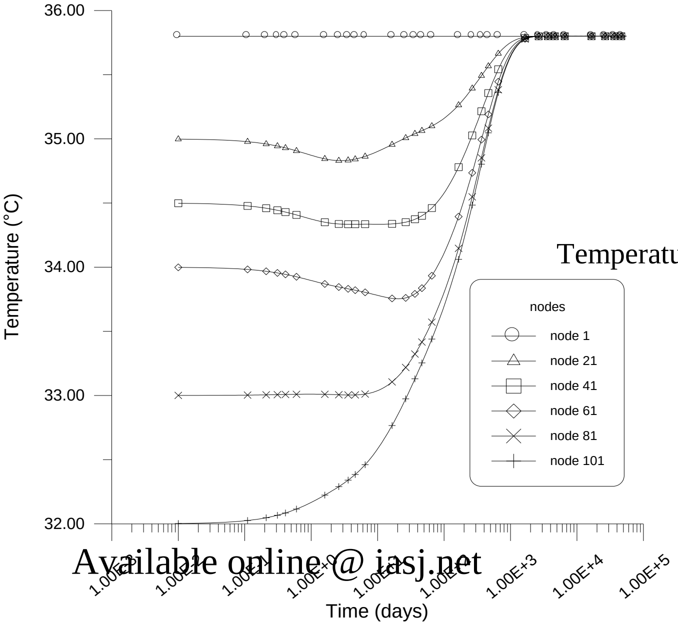
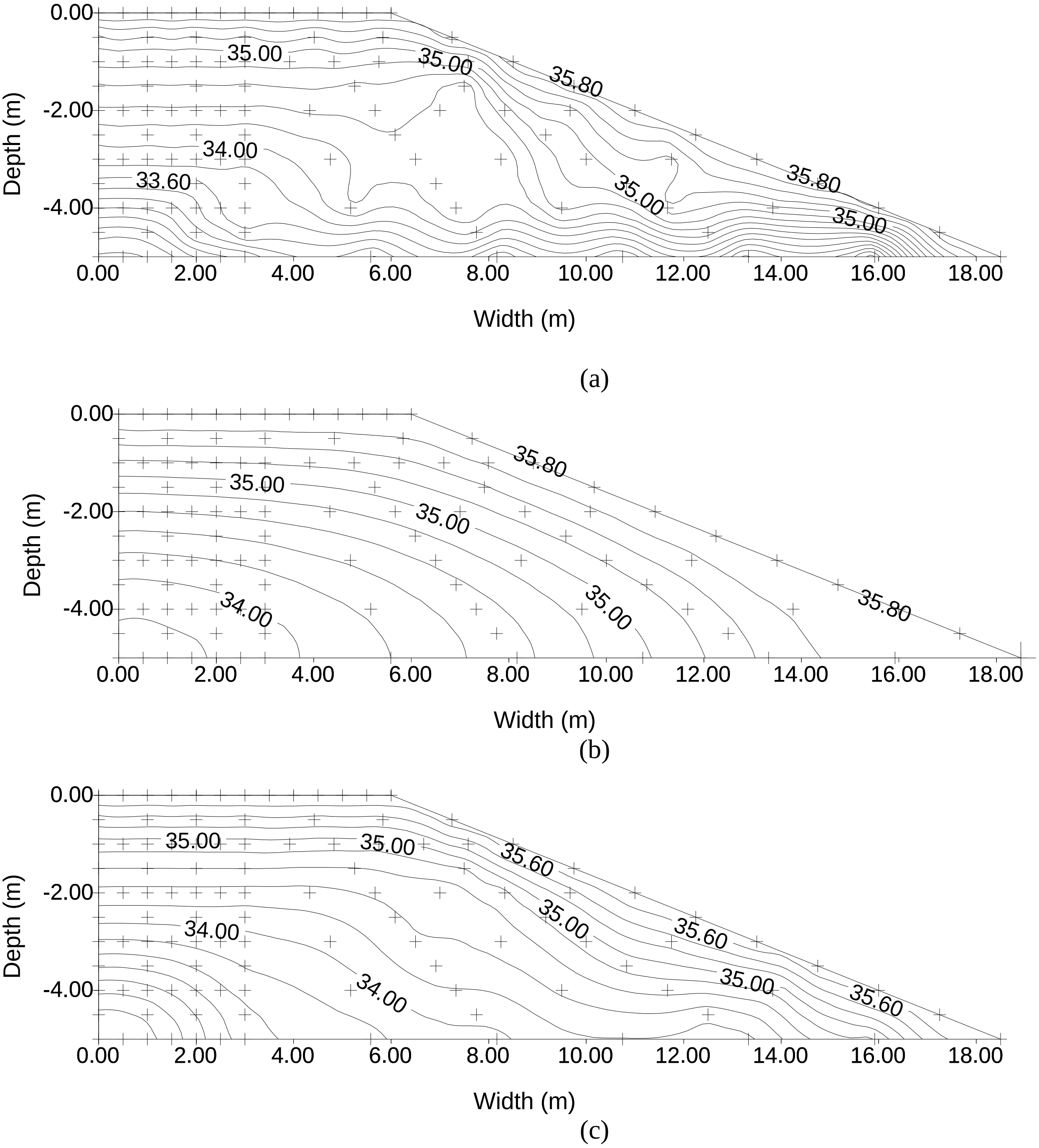


Fig 7

Temperature variation with time for different nodes at the center line early stages of consolidation. This stage is followed by a descent in the rate of settlement due to air propagation through the soil, then after about 5 to 7 days, there is an increase in the rate of settlement and the consolidation process continues as usual.

The settlement may take longer time than expected due to the lack of enough loads that could accelerate the consolidation process.

Finally, this analysis should not be compared with any other analysis made before where the Mdaina embankment was assumed fully saturated there. This is due to that some of the parameters used in this analysis are assumed for demonstrative purposes.



(c)

Fig 8

Temperature distribution in the embankment

a) after 1 day ; b) after 10 days; c) after 100 days

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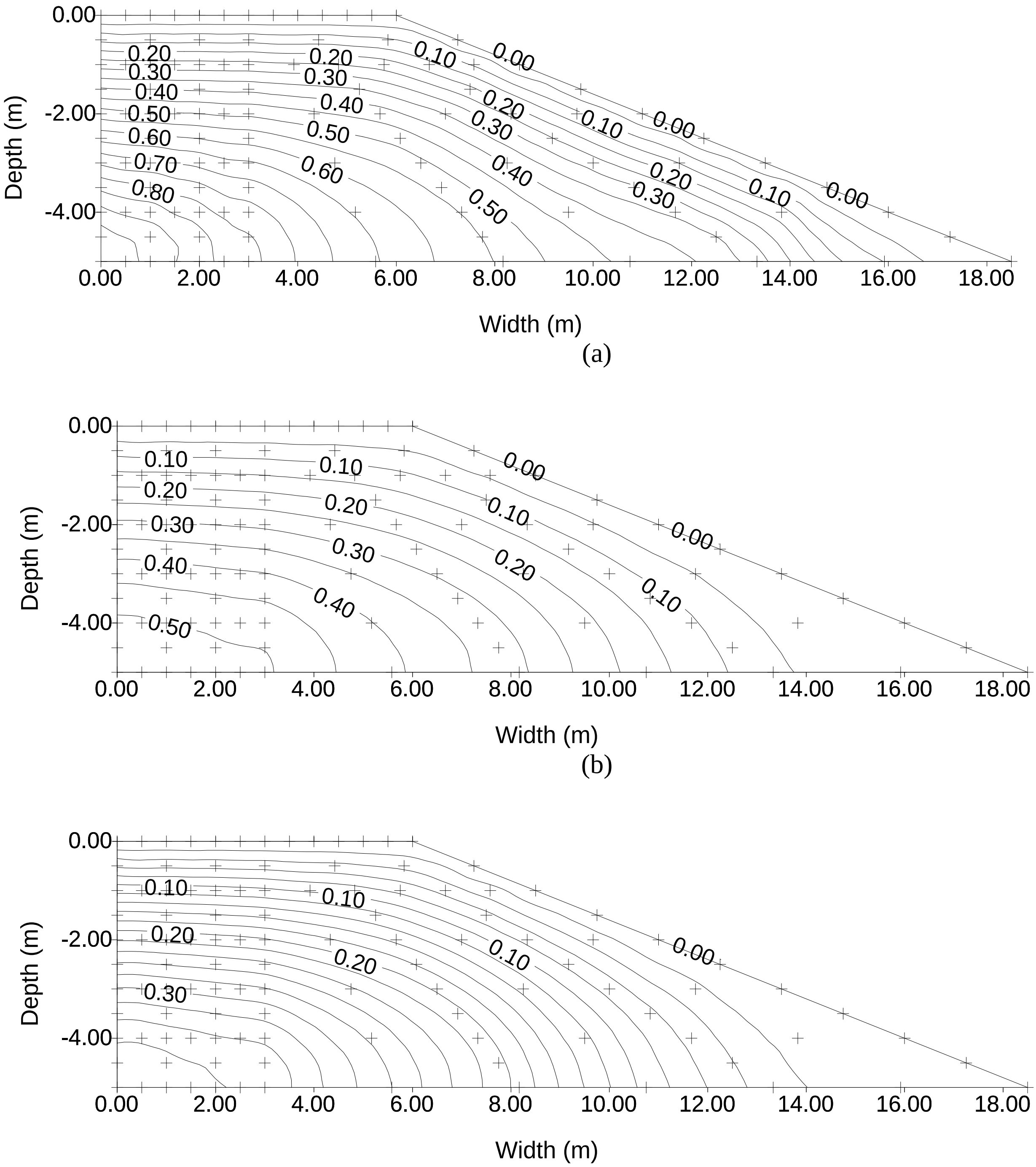


(a)

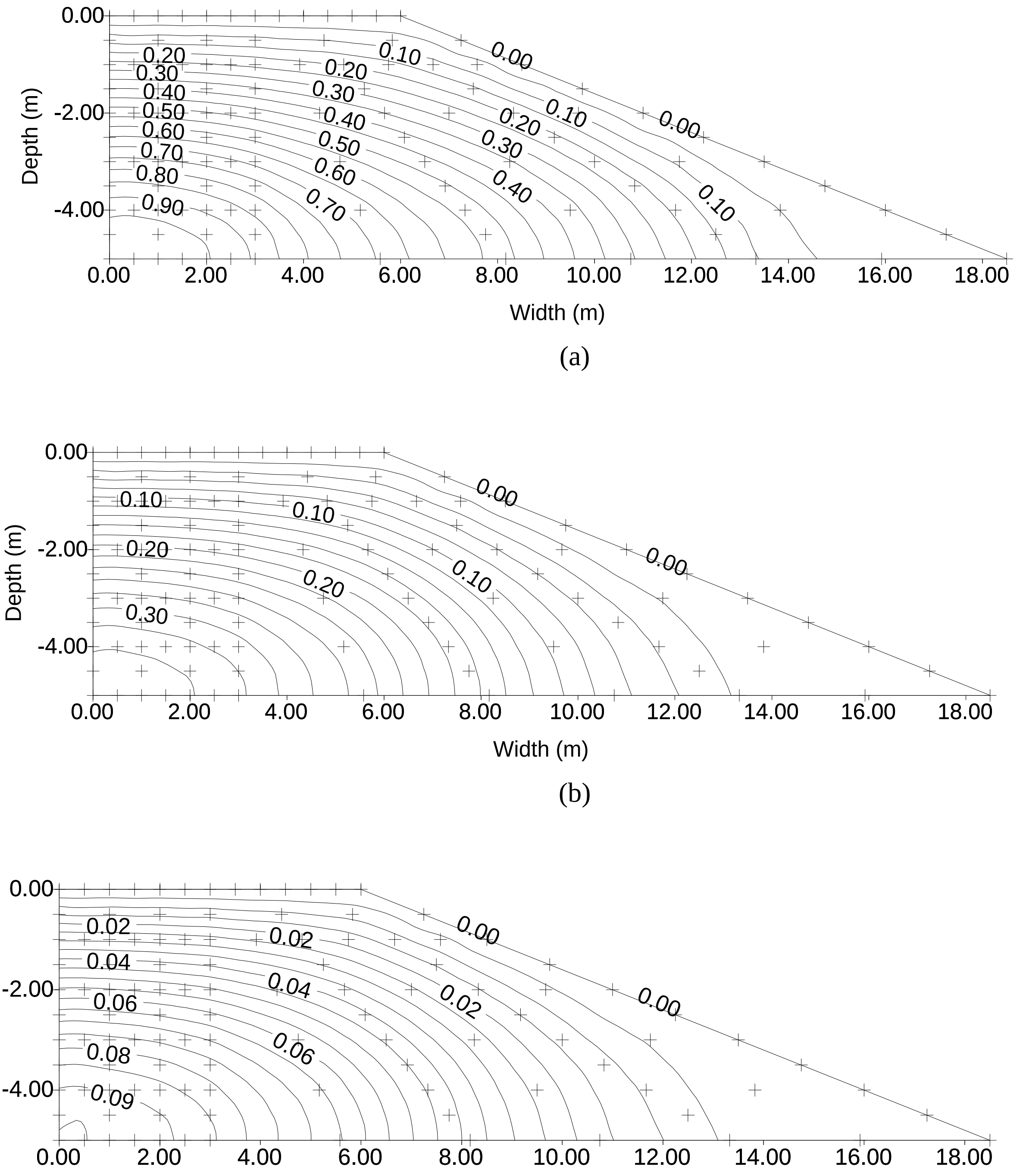
(b)

(c)

Fig 9

Water pressure distribution, contour lines represent u_w/u_{wmax}

a) at T_{vw}=0.1 ; b) at T_{vw}=0.5 ; c) at T_{vw}=0.9



(a)

(b)

(c)

Fig 10

Air pressure distribution, contour lines represent u_a/u_{amax}

a) at T_{va}=0.1 ; b) at T_{va}=0.5 ; c) at T_{va}=0.9

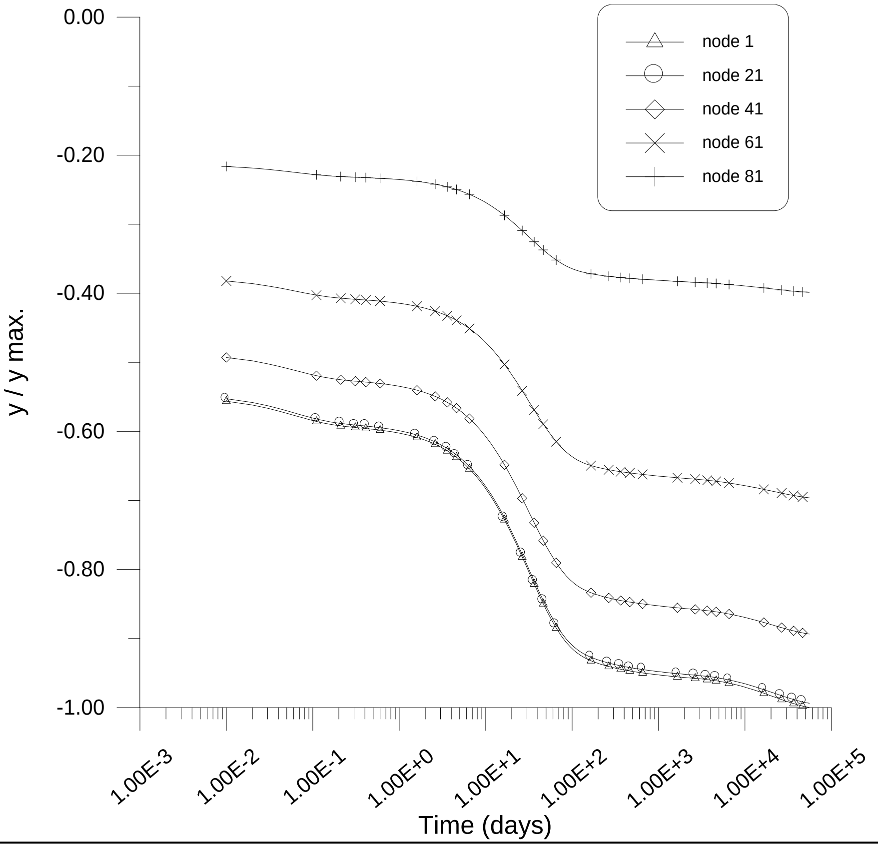


Fig11
Vertical displacement for various nodes at the center line of the embankment

CONCLUSIONS

As the subject of modeling of partially saturated soils behavior is considered relatively new, many concluding ideas can be formed in different aspects regarding this regime.
From the analysis and discussion of the consolidation problem for the embankment under its self weight, it is concluded that: –

1. The water–phase pressure dissipation behavior is the same as that for the classical consolidation process for ordinary loading.
2. The pore–air pressure dissipation takes the normal shape of the classical consolidation process. This was observed for a degree of saturation about 95%.
3. The vertical displacement for the partially saturated consolidation is nearly similar to the vertical displacement for the fully saturated condition. Except that after the first region (about 1 or 2 days) the vertical displacement begins to accelerate due to pore–air pressure dissipation.
4. Prediction of degree of consolidation is not due to pore–water pressure dissipation only, but pore–air pressure dissipation should be considered also. The degree of consolidation should be calculated separately for the water–phase and the air–phase.

Generally, the increase of temperature causes an increase in the pore–air and pore–water pressures and vice versa. This increase in pressure may cause an expansion in the soil skeleton, and may cause both pressures to take a longer time to dissipate.

The effect of soil skeleton expansion is neglected due to its small effect for many types of soils. If it is taken into consideration, it may affect the soil skeleton and cause more expansion in it.

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APPENDIX A

Definitions of matrices

$$[\mathbf{K}] = -\int_{\Omega} [\mathbf{B}]^{\mathbf{T}}[\mathbf{D}][\mathbf{B}]dxdy \qquad \text{...(A - 1)}$$

$$[\mathbf{L}] = \left(n_w - \frac{m_2^s}{m_1^s} + n_s \right) \int_{\Omega} \mathbf{B}^{\mathbf{T}}\mathbf{m}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{...(A - 2)}$$

$$[\mathbf{M}] = \left(\frac{m_2^s}{m_1^s} - n_w \right) \int_{\Omega} \mathbf{B}^{\mathbf{T}}\mathbf{m}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{...(A - 3)}$$

$$\mathbf{f} = \frac{\partial}{\partial t}\left(\int_{\Omega} [\mathbf{N}]^{\mathbf{T}}[\mathbf{BF}]dxdy - [\mathbf{BC}] \right) \qquad \text{...(A - 4)}$$

$$\mathbf{W1} = \frac{m_1^w}{m_1^s} \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{mB}dxdy \qquad \text{....(A - 5)}$$

$$\mathbf{W2} = \left(m_2^w - \frac{m_1^w m_2^s}{m_1^s} \right) \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{....(A - 6)}$$

$$\mathbf{W3} = \left(\frac{m_1^w m_2^s}{m_1^s} - m_2^w \right) \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{....(A - 7)}$$

$$\mathbf{KW} = -\int_{\Omega} \mathbf{R}^{\mathbf{T}}[\mathbf{DW}]\mathbf{R}dxdy \qquad \text{....(A - 8)}$$

$$\mathbf{UT} = -\int_{\Omega} \mathbf{R}^{\mathbf{T}}[\mathbf{DT}]\mathbf{R}dxdy \qquad \text{....(A - 9)}$$

$$\mathbf{A1'} = \frac{\left(m_1^s - m_1^w\right)}{m_1^s} \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{mB}dxdy \qquad \text{....(A - 10)}$$

$$\mathbf{A2'} = \frac{\left(m_1^w m_2^s - m_1^s m_2^w\right)}{m_1^s} \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{....(A - 11)}$$

$$\mathbf{A3'} = \frac{\left(m_1^s m_2^w - m_1^w m_2^s\right)}{m_1^s} \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{....(A - 12)}$$

$$\mathbf{KAUa} = -\int_{\Omega} \mathbf{R}^{\mathbf{T}}[\mathbf{DAUa}]\mathbf{R}dxdy \qquad \text{....(A - 13)}$$

$$\mathbf{KAT} = \int_{\Omega} \mathbf{R}^{\mathbf{T}}[\mathbf{DAT}]\mathbf{R}dxdy \qquad \text{....(A - 14)}$$

$$\mathbf{FA'} = -\frac{Dh}{\bar{u}_{ao}} \oint_{S} N_i \frac{\partial N_i}{\partial n}\mathbf{u_a}dS + \frac{Dh}{T_o} \oint_{S} N_i \frac{\partial N_i}{\partial n}\mathbf{T}dS \qquad \text{....(A - 15)}$$

$$[\mathbf{DH}] = \begin{bmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{bmatrix} \qquad \text{....(A - 16)}$$

$$[\mathbf{T1}] = \zeta \int_{\Omega} \mathbf{N}^{\mathbf{T}}\mathbf{N}dxdy \qquad \text{....(A - 17)}$$

$$[\mathbf{KH}] = \int_{\Omega} \mathbf{R}^{\mathbf{T}}[\mathbf{DH}]\mathbf{R}dxdy \qquad \text{....(A - 18)}$$

$$[\mathbf{FH}] = \int_{\Omega} \left(N_i \frac{\partial N_j}{\partial x}\lambda_x + N_i \frac{\partial N_j}{\partial y}\lambda_y \right)\mathbf{T}dxdy \qquad \text{....(A - 19)}$$

$$\mathbf{B}^{\mathbf{T}} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial y} \\ 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \qquad \text{....(A - 20)}$$

$$\mathbf{D} = \begin{bmatrix} \frac{2(1-\mu)}{m_1^s} & \frac{2\mu}{m_1^s} & 0 \\ \frac{2\mu}{m_1^s} & \frac{2(1-\mu)}{m_1^s} & 0 \\ 0 & 0 & G \end{bmatrix} \qquad \text{....(A - 21)}$$

APPENDIX B

Input Data Required for Partially Saturated Soils

In addition to the input data required for the usual case problems, there are some additional parameters that are needed in dealing with partially saturated soil characteristics, which are:–

- **Volumetric Coefficient of Solubility, h*:***
  which is defined as: –

$$h = (\rho_w/\rho_a)\text{H} \qquad \text{(B-1)}$$

where
H is the coefficient of solubility, and
h for air varies with temperature as shown in figure B-1.

An extrapolation is made to the available data using an exponential function best fit curve obtained by using the computer program Wingraph 1.09[**]. The equation obtained has the following form: –

$$y = \exp(-0.0206841 * x) * 0.0285497 \qquad \text{(B-2)}$$

This equation is included in the computer program to obtain the volumetric coefficient of solubility at each temperature.

- **Coefficient of Diffusion D:**
  Its value is taken as 2.0×10^{-9} m^2/s (Fredlund and Rahardjo, 1993).

- **Coefficients of Volume Change:**
  The values of m_1^s, m_2^s, m_1^w, and m_2^w are taken as 0.006×10^{-4} kPa^{-1}, 0.032×10^{-4} kPa^{-1}, 0.131×10^{-4} kPa^{-1}, and 0.657×10^{-4} kPa^{-1}, respectively (Fredlund and Rahardjo, 1986). These values are taken as results from experimental tests. True data must be obtained from laboratory experiments.

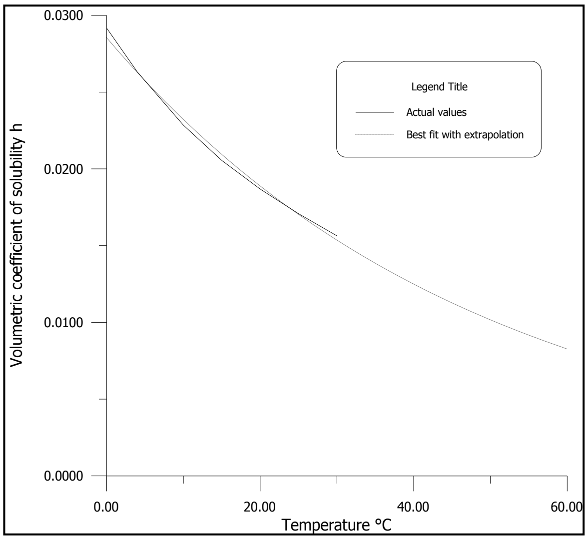

Fig. B - 1
Variation of coefficient of solubility with temperature showing Data extrapolation (after Fredlund, and Rahardjo, 1993)

[**] The computer program Wingraph Ver. 1.09 is a software produced by Golden Software, Inc. 1993 and works under WINDOWS 95/98/2000/*Me*.