MODELLING OF IRAQI GYPSEOUS SOIL BEHAVIOUR UNDER STRESS-FLOW-DISSOLUTION CONDITIONS

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ABSTRACT

Gypseous soils are distributed in many regions in Iraq. Therefore, it is necessary to study the geotechnical properties of such soils due to the possible large damage that may incur structures founded and constructed in or on them. The soil used in this study is from Al-Najef City, Iraq. It is poorly graded sand. It also has a gypsum content of about 28-32٪. The mineralogical and chemical properties of the soil are established at first. However, this study is concerned with the dissolution of gypsum and its effect on the soil. The importance of the progress of dissolution is verified through the study of the characteristics of the soil skeleton and the pore fluid. Three differential equations are used to study this effect, namely, *continuity*, *equilibrium* and *dispersion*. They are solved by using the finite element method. In addition, this work uses the hyperbolic stress-strain idealization as a constitutive relationship. Tri-axial (CD) tests are conducted to find the hyperbolic parameters. An experimental setup is modified to find the longitudinal and lateral coefficients of dispersion. One- and two-dimensional problems are solved to study the effect of dissolution. Results reveal high effects of dissolution of gypsum on the settlement, pore water pressure, elastic modulus and Poisson's ratio values. Settlement increases while other parameters (E, B and υ) decrease with increasing dissolution. Furthermore, there is a vast behavioral difference between one and two dimensional problems.

الخلاصة

تنتشر الترب الجبسية في مناطق عديدة من العراق. لذلك من الضروري دراسة الخواص الفيزياوية لهذه الترب بسبب الاضرار المحتملة الكبيرة التي تحدث بالمنشاءآت نتيجة لبناءها عليها او بداخلها. لقد استخدمت تربة من هذا النوع تم جلبها من مدينة النجف في العراق. ان نوع التربة هو رمل متدرج بشكل سيء كما ان لها محتوى من الجبس قدره حوالي **28–32%**. لقد تم التحقق من الخواص المعدنية و الكيمياوية للتربة اولاَ. بعد ذلك, تتطرق هذه الدراسة الى اذابة الجبس و تأثيره على التربة. لقد تم التحقق من تقادم الاذابة من خلال دراسة خواص الهيكل الصلب و مائع المسام. لقد تم استخدام ثلاثة معادلات تفاضلية لدراسة هذه الظاهرة و هي : الاستمرارية و الاتزان و الانتشار. لقد تم حل هذه المعادلات باستخدام طريقة العناصر المحددة. اضافة الى ذلك, تم استخدام تمثيل اجهاد– انفعال القطع الناقص

كعلاقة تكوينية. لقد تم اجراء فحوص ثلاثية محاور مبزولة لايجاد معاملات علاقة القطع الناقص. لقد تم تعديل جهاز لايجاد معاملي الانتشار الطولي و الجانبي. لقد تم حل مسألتين ببعد واحد و ببعدين لدراسة خواص الاذابة. تظهر النتائج تأثيرات عالية لاذابة الجبس على الهبوط و ضغط الماء المسامي و قيم معاملي المرونة و بويزون. يزداد الهبوط بينما تقل قيم المعاملات الاخرى (معاملي المرونة و التضخم و نسبة بويزون) مع زيادة الاذابة.تظهر الدراسة فرقاَ كبيراَ في الخواص عند دراسة المسألة ببعدين عن البعد الواحد.

KEY WORDS
Gypseous, Dissolution, Dispersion, Finite element, Hyperbolic.

INTRODUCTION

The term "gypsiferous soil" and “gypseous soil” are used to specify the soil that contains gypsum, the first is used by agronomists, while the second is used by civil engineers. Gypsum is present in soils in the form of calcium sulphate dihydrate ($CaSO_4.2H_2O$). A transitional form of calcium sulphate ($CaSO_4.0.5H_2O$) is sometimes found at the soil surface in extremely dry climates (Doner and Lynn, 1977). There is no unique definition for gypseous soils used by civil engineers. It can be stated that a gypseous soil is one in which has a gypsum content enough to change or to affect its engineering properties. Gypseous soils are distributed in many regions in the world including Iraq. They cover about (20%) of Iraq's area. Gypseous soils in Iraq cover about 7.28% of the gypseous soils in the world and 16.2% from Asia (FAO, 2001). Many problems relating to construction on gypseous soils were observed. There are three main sources of these problems; first, the *dissolution* and *transportation* of gypsum through soil causes a continuous loss of soil mass and increasing voids. A large reduction in shear strength and an increase in compressibility are the main results of this phenomenon. The second is the variation of shear strength and compressibility characteristics of gypseous soils upon wetting and saturation. The third is the volume change accompanying the dehydration of gypsum or hydration of anhydrite. In the first case, a volume decrease of approximately 39% may be reached, while in the second case, the volume may be increased by 63%)Ismail-1993). The main purpose of this study is to simulate the mechanical behavior of gypseous soils before and during leaching by using the *finite element method* for solving three differential equations, namely, *dispersion* , *flow* and *continuity*. In addition to that, set up experimental facilities which are used for the determination of the dispersive characteristics of gypseous soils (Al-Hassanee, 2004).

LEACHING STRAIN

Consider the element shown in Figure (1) which has a volume (V_o) and the water flows through the element having a velocity (v_i). This element contains some gypsum. Due to flow of water, the gypsum dissolves and causes a decrease in volume: -

$$d_{mass} = dCV_w \quad (1)$$

$$d_{mass} = dC\, v_i\, A\, \Delta t \quad (2)$$

where: -

C: concentration of gypsum in water (mg/l),
t: time (sec),

v_i: velocity (m/sec),
V_w: volume of water (L^3), and

A: area of section (m^2).

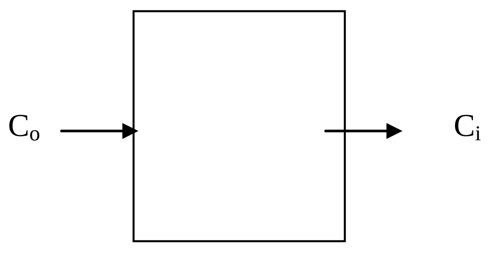


Fig (1) Variation of concentration through element

$$d\,volume = \frac{dC(v_i A \Delta t)}{G_g \gamma_w} \quad (3)$$

where:
G_g: specific gravity of gypsum (with an average value of 2.32),
γ_w : density of water (kg/m^3),

Leaching strain $= dC\left(\dfrac{v_i A \Delta t}{G_g \gamma_w V_o}\right)$, and (4)

V_o: initial volume of element (L^3).

CONSTITUTIVE EQUATION

The constitutive equation relating effective stress ($\bar{\sigma}$) to the strain of the skeleton is independent of the pore pressure and for a general non- linear material, it can be written in a tangential form thus allowing plasticity, if desired, to be incorporated. If leaching strain is present, the expression is written in a general form as (Lewis and Schrefler, 1987): -

$$d\bar{\sigma} = D_T\left(d\varepsilon - d\varepsilon p - d\varepsilon_m - d\varepsilon_o\right) \qquad (5)$$

where:

$$d\varepsilon p = -m\left(\frac{dp}{3K_s}\right) \qquad (6)$$

$$d\varepsilon_m = dCV_w G_g \delta_w v_o \qquad (7)$$

in which: -

D_T : is the tangent matrix,
m : is equal to unity for the normal stress components and zero for shear stress components,
$d\varepsilon$: the total strain of skeleton,
$d\varepsilon p$: the overall volumetric strain caused by uniform compression of the particles due to the pressure of the pore fluid,
K_s : the bulk modulus of the solid phase,
$d\varepsilon_m$: the strain due to the dissolved mass of calcium sulphate,
$d\varepsilon_o$: represents all other strains not directly associated with stress changes (swelling, thermal, etc.); (Zienkiewicz, 1977), and
dp : is the pore water pressure.

The equilibrium equation relating the total stress (σ) to the body forces (b) and the boundary traction ($\hat{t}$) specified at the boundary (Γ) of the domain (Ω) is formulated in terms of the unknown displacement vector (u). Using the *principle of virtual work*, the general equilibrium statement can be written as (Zienkiewicz, 1977): -

$$\int_\Omega \delta\varepsilon^T \sigma d\Omega - \int_\Omega \delta u^T b d\Omega - \int_\Gamma \delta u^T \hat{t} d\Gamma = 0 \qquad (8)$$

Furthermore, upon taking into account the constitutive relationship given by Equation (5) and dividing by dt, the following equation is obtained: -

$$\int_\Omega \delta\varepsilon^T D_T\left(\frac{\partial\varepsilon}{\partial t}\right)d\Omega - \int_\Omega \delta\varepsilon^T m\left(\frac{\partial p}{\partial t}\right)d\Omega + \int_\Omega \delta\varepsilon^T\left(\frac{\partial p}{\partial t}\right)\left(\frac{1}{3K_s}\right)d\Omega -$$

$$\int_\Omega \delta\varepsilon^T D_T\left(\frac{\partial C}{\partial t}\right)\left(\frac{V_w}{G_g \gamma_w V_o}\right)d\Omega - \int_\Omega \delta\varepsilon^T D_T\left(\frac{\partial\varepsilon_o}{\partial t}\right)d\Omega - \left(\frac{df}{dt}\right) = 0 \qquad (9)$$

DARCY'S LAW (FLUID PHASE)

The geometrical complexity of a porous medium renders impossible a strict analytical treatment of the fluid velocity within the porous space. To overcome this obstacle, the fictitious seepage velocity (also known as *bulk* or *Darcy's velocity*) is defined as (Lewis and Schrefler, 1987): -

$$q = -\left(\frac{1}{\mu}\right)k\nabla\left(p + pgh\right) \tag{10}$$

where:

k : is the absolute permeability matrix of the medium,
μ : the dynamic viscosity of the fluid,
p : the fluid pressure,
ρ : the density of the fluid,
g : the gravitation acceleration, and
h : total head.

CONTINUTY EQUATION

The continuity of flow requires that the following expression is valid (Crichlow, 1977):

Rate of fluid accumulation = $+\nabla(\rho g) = 0$ (11)

which upon combining with Darcy's law given by Equation (10) results in: -

- rate of fluid accumulation = $+\nabla\left\{-\left(\frac{k\rho}{\mu}\right)\nabla\left(p + \rho gh\right)\right\} = 0\cdot$ (12)

There are many factors which contribute to the rate of fluid accumulation and these are enumerated as follows (Lewis et al., 1976): -

a. Rate of change of total strain $\left(\frac{\partial \varepsilon_v}{\partial t}\right) = m^T\left(\frac{\partial \varepsilon}{\partial t}\right)$.

b. Rate of change of the soil volume due to pressure change = $\left(\frac{1-n}{K_s}\right)\left(\frac{\partial p}{\partial t}\right)$.

c. Rate of change of saturation = $n\rho\left(\frac{\partial S}{\partial t}\right)$.

d. Rate of change of fluid density = $nS\left(\frac{\partial \rho}{\partial t}\right)$.

e. Change of soil size due to effective stress change = $\left(\frac{\partial \sigma}{\partial t}\right) - \left(\frac{1}{3K_s}\right)m^T\left(\frac{\partial \sigma}{\partial t}\right)$.

The continuity equation for water, therefore, becomes: -

$$-\nabla^T\left\{k\nabla\left(p + \rho gh\right)\right\} + \left(m^T - \left(\frac{m^T D_T}{3K_s}\right)\right)\left(\frac{\partial \varepsilon}{\partial t}\right) + \left(\frac{m^T D_T}{3K_s}\right)\left(\frac{\partial C}{\partial t}\right)\left(\frac{V_w}{G_g \gamma_w V_o}\right) + \left[\left(\frac{1-n}{K_s}\right) - \left(\frac{1}{3K_s}\right)^2\left(m^T D_{Tm}\right)\right]\left(\frac{\partial p}{\partial t}\right) = 0 \tag{13}$$

where:

$$K = k\left(\frac{\rho g}{\mu}\right) \quad (14)$$

and is known as the *coefficient of permeability* or *hydraulic conductivity* matrix whose coefficients have units of length over time.

DERIVATION OF THE ADVECTION-DISPERSION EQUATION

An important relationship in fluid flow is the *principle of conservation of mass*. This principle is a statement of material balance with respect to a volume element fixed in space and may be simply stated as: -

(Rate of Mass Outflow – Rate of Mass Inflow)= (Rate of Change of Mass Inside the volume element). (15)

Applying this principle to the volume element shown in Figure (2) results in: -

$$M_{(X_1+\Delta X_{1/2})} - M_{(X_1-\Delta X_{1/2})} + M_{(X_2+\Delta X_{2/2})} - M_{(X_2-\Delta X_{2/2})} + \quad (16)$$

(4-1)

$$M_{(X_3+\Delta X_{3/2})} - M_{(X_3-\Delta X_{3/2})} = Mp - \frac{\partial M_{VE}}{\partial t}$$

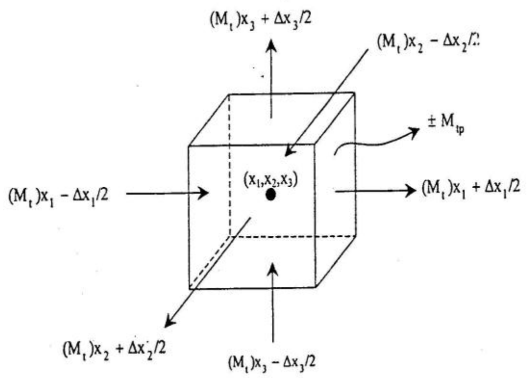


Fig (2*): Representative elementary volume of porous medium used to develop continuity equation for gypsum in miscible fluid flow*

where: -

$M_{(X_1+\Delta X_{1/2})}$, $M_{(X_2+\Delta X_{2/2})}$, $M_{(X_3+\Delta X_{3/2})}$ = rate of mass out–flow across faces $x_1 + \Delta x_{1/2}, x_2 + \Delta x_{2/2}, x_3 + \Delta x_{3/2}$, and

$M_{(X_1-\Delta X_{1/2})}$, $M_{(X_2-\Delta X_{2/2})}$, $M_{(X_3-\Delta X_{3/2})}$ = rate of mass in-flow across faces $x_1 - \Delta x_{1/2}, x_2 - \Delta x_{2/2}, x_3 - \Delta x_{3/2}$.

M_{VE} = mass contained inside the volume element,

MP = mass source or sink term which is positive \
when a sink.

Applying a *Taylor series expansion* about the point X_1, X_2 and X_3, neglecting second and higher order terms and then substituting into Equation (16), the following is obtained after each one of the mass flow rate components is expressed in terms of the fluid density, the dimensions of the volume element and the volume flux:

$$M_{xi} = \rho q_i.\Delta x_i.\ \Delta x_j \quad (17)$$

$$M_{VE} = \rho \Phi S.\Delta x_1.\Delta x_2.\Delta x_3 \quad (18)$$

$$MP = \rho p Q \quad (19)$$

Equation (15) gives:

$$\frac{\partial}{\partial x_1}(\rho q1.\Delta x_2.\Delta x_3)\Delta x_1 + \frac{\partial}{\partial X_2}(\rho q2.\Delta x_1.\Delta x_3)\Delta x_2 + \frac{\partial}{\partial x_3}(\rho q3.\Delta x_1.\Delta x_2)\Delta x_3 = \rho p Q - \frac{\partial}{\partial t}(\rho \Phi S.\Delta x_1.\Delta x_2.\Delta x_3) \quad (20)$$

Reddle and Sunada (1974) used Fick’s law for describing diffusion on a microscopic scale in a porous medium. The following is obtained: -

$$\hat{C}\hat{V}_{ti} = \hat{C}V - Dd\hat{T}_{ij}\left(\frac{\hat{C}}{x_j}\right) \tag{21}$$

$\hat{C}$: concentration of tracer (gypsum in our case) in fluid element ,

$\vec{\hat{V}}_t$: velocity of the tracer in fluid element with respect to a fixed coordinate system,

$\vec{\hat{V}}$: volumetric velocity of fluid element , and

Dd : is the coefficient of molecular diffusion.

By using macroscopic analysis and assuming isothermal conditions gives (Reddle and Sunada, 1974):

$$\left(\frac{\partial C}{\partial t}\right) = \left(\frac{\rho}{(\rho - \alpha C)}\right)\left(\frac{DD}{\Phi S \Delta x_1 \Delta x_2 \Delta x_3}\right) - V_i\left(\frac{\partial C}{\partial X_i}\right) + (Cp - C)\left(\frac{Q}{\Phi S \Delta x_1 \Delta x_2 \Delta x_3}\right) + \left(\frac{\rho_o C \beta}{(\rho - \alpha C)}\right)\left(\left(\frac{\partial P}{\partial t}\right) + V_i\left(\frac{\partial P}{\partial x_i}\right)\right) \tag{22}$$

where,

β : fluid compressibility,

α : proportionality factor relating C and ρ,

P_o : is the original value of pressure,

ρ_o : is the original value of density,

C_o : is the original value of concentration, and

O : is the original value of the variable (Reddle and Sunada, 1974).

The following assumptions are made for simplifying the dispersion equation: -

1. The volume element is completely saturated with water (S = 1).
2. The porous medium is homogeneous so that the porosity will be independent of position.
3. The density does not vary with concentration ($\alpha = 0$).

Accordingly, Equation (22) becomes:

$$\left(\frac{\partial C}{\partial t}\right) = \left(\frac{\partial}{\partial X_i}\right)\left[\left(D_{ij} + DdT_{ij}\right)\left(\frac{\partial C}{\partial X_i}\right)\right] - V_i\left(\frac{\partial C}{\partial X_i}\right) + (Cp - C)\left(\frac{Q}{\Phi S \Delta x_1 \Delta x_2 \Delta x_3}\right) + C\beta\left(\frac{\partial P}{\partial t}\right) + CV_i\left(\frac{\partial P}{\partial x_i}\right) \tag{23}$$

$$D_{ij} = D_{22}\delta_{ij} + (D_{11} - D_{22})\frac{V_i V_j}{V} \tag{24}$$

Expanding the advection – dispersion equation from equation (24) for a two–dimensional flow, and considering that the transport by molecular diffusion is negligible, the Equation (23) will be as follows: -

$$\left(\frac{\partial C}{\partial t}\right) = D_{11}\left(\frac{\partial C^2}{\partial x^2}\right) + D_{22}\left(\frac{\partial C^2}{\partial y^2}\right) - V_x\left(\frac{\partial C}{\partial x_i}\right) + (Cp - C)\left(\frac{1}{nV_o}\right)\left(V_x A_y + V_y A_x\right) + C\beta\left(\frac{\partial P}{\partial t}\right) + CV_i\left(\frac{\partial P}{\partial x_i}\right) \tag{25}$$

FINITE ELEMENT APPLICATION

The fully coupled solution of the one-phase flow (α_1–species) equation in a non-linear porous medium will now be presented in detail. The particular form of the *equilibrium equation* (9), together with the *continuity equation* (13) and the *advection-dispersion equation* (25) represent the governing equations for soil mechanics problems within the lines of Biot's self-consistent theory. The first two mentioned equations are re-written again for the sake of completeness: -

$$\int_\Omega \delta\varepsilon^T D_T\left(\frac{\partial\varepsilon}{\partial t}\right)d\Omega - \int_\Omega \delta\varepsilon^T m\left(\frac{\partial p}{\partial t}\right)d\Omega + \int_\Omega \delta\varepsilon^T D_T m\left(\frac{\partial p}{\partial t}\right)\left(\frac{1}{3K_s}\right)d\Omega -$$
$$\int_\Omega \delta\varepsilon^T D_T\left(\frac{\partial C}{\partial t}\right)\left(\frac{V_w}{G_s\gamma_w V_o}\right)d\Omega - \int_\Omega \delta\varepsilon^T D_T\left(\frac{\partial\varepsilon_o}{\partial t}\right)d\Omega - \left(\frac{df}{dt}\right) = 0 \qquad (26)$$

$$\overline{A} = -\nabla^T\left\{\frac{k}{\mu}\nabla(P+\rho gh)\right\} + \left(m^T - \left(\frac{m^T D_T}{3K_s}\right)\right)\left(\frac{\partial\varepsilon}{\partial t}\right) +$$
$$\left(\frac{m^T D_T}{3K_s}\right)\left(\frac{\partial C}{\partial t}\right)\left(\frac{V_w}{G_g\gamma_w\partial_o}\right) + \left[\left(\frac{1-n}{K_s}\right) - \left(\frac{1}{3K_s}\right)^2\left(m^T D_{Tm}\right)\right]\left(\frac{\partial p}{\partial t}\right) = 0 \qquad (27)$$

$$B = -n^T\frac{k}{\mu}\nabla(p+\rho gh) - q = 0 \qquad (28)$$

By using Green's theorem (26) and (27) become (Zienkiewicz, 1977):

$$\int_\Omega\left\{\nabla a^T\left(\frac{k}{\mu}\right)\nabla(p+\rho gh) + a^T\left[m^T - \left(\frac{m^T D_T}{3K_s}\right)\left(\frac{\partial\varepsilon}{\partial t}\right) + \left(\frac{m^T D_T}{3K_s}\right)\left(\frac{\partial C}{\partial t}\right)\left(\frac{V_w}{G_s\gamma_w V_o}\right) + \right.\right. \qquad (29)$$
$$\left.\left.\left[\left(\frac{1-n}{K_s}\right) + \left(\frac{n}{K_w}\right) - \left(\frac{1}{(3K_s)^2}\right)m^T D_T m\right]\left(\frac{\partial p}{\partial t}\right)\right]\right\}d\Omega - \int_\Gamma\left\{a^T n^T\left(\frac{k}{\mu}\right)\nabla(p+\rho gh) + b^T n^T\left(\left(\frac{k}{\mu}\right)\nabla(p+\rho gh)\right) + b^T q\right\}d\Gamma = 0$$

$$\int_\Omega\left\{\nabla a^T\left(\frac{k}{\mu}\right)\nabla(p+\rho gh) + a^T\left[m^T + \left(\frac{n}{K_w}\right)\left(\frac{\partial p}{\partial t}\right)\right]\right\}d\Omega - \int_\Gamma a^T q\,d\Gamma = 0 \qquad (30)$$

The *Galerkin method* is applied where a = N. The finite element discretization gives the result (Zienkiewicz and Morgan, 1982):

$$\delta u^T\left\{\int_\Omega B^T D_T B d\Omega\left(\frac{d\bar{u}}{dt}\right) - \int_\Omega B^T m\overline{N}d\Omega\left(\frac{d\bar{p}}{dt}\right) - \int_\Omega B^T D_T\left(\frac{v_i A\Delta t}{G_g\gamma_w V_o}\right)Nd\Omega\left(\frac{d\overline{C}}{dt}\right) - \right.$$
$$\left.\left(\frac{1}{dt}\right)\int_\Omega B^T D_T d\varepsilon_o d\Omega\right\} - \sigma u^T\left\{\int_\Omega N^T\left(\frac{db}{dt}\right)d\Omega + \int_\Gamma N^T\left(\frac{dt'}{dt}\right)\right\} = 0 \qquad (31)$$
$$\int_\Omega(\nabla N)^T\left(\frac{k}{\mu}\right)\nabla\overline{N}d\Omega\bar{p} + \int_\Omega N^T m^T B d\Omega\left(\frac{d\bar{u}}{dt}\right)\int_\Omega N^T\left(\frac{n}{K_w}\right)\overline{N}d\Omega\left(\frac{dp}{dt}\right) + \int_\Omega N^T\nabla^T\left(\frac{k}{\mu}\right)\nabla\rho gh d\Omega\int_\Gamma N^T q d\Gamma = 0$$

As for the third differential equation for advection-dispersion (Equation 25), assuming a trial function of the form (Zienkiewicz and Morgan, 1982):

$$C \approx \hat{C} = C_I(t)N_I(x_i)P_I(t) \qquad (32)$$

Applying the weighted residual method with Galerkin's method and integrating the second spatial derivative term by using Green's theorem (Zienkiewicz, 1977), Equation (25) becomes: -

$$\iint\Bigg[Dxx\left(\frac{\partial N_I}{\partial x}\right)\left(\frac{\partial N_J}{\partial x}\right)C_J + Dyy\left(\frac{\partial N_I}{\partial y}\right)\left(\frac{\partial N_J}{\partial y}\right)C_J + VxN_I\left(\frac{\partial N_J}{\partial x}\right)C_J +$$
$$VyN_I\left(\frac{\partial N_J}{\partial x}\right)C_J + N_I N_J\left(\frac{\partial C_J}{\partial t}\right) + C\beta N_I N_J\left(\frac{\partial P_J}{\partial t}\right) + CVxN_I\left(\frac{\partial N_J}{\partial x}\right)P_J + \quad (33)$$
$$CVyN_I\left(\frac{\partial N_J}{\partial y}\right)P_J + NV_x(Cp-C)\left(\frac{Ax}{nV_o}\right) + NV_y(Cp-C)\left(\frac{Ay}{nV_o}\right)\Bigg]dxdy - \int_\Gamma\left[Dxx\left(\frac{\partial \hat{C}}{\partial x}\right)I_x + Dyy\left(\frac{\partial \hat{C}}{\partial y}\right)I_y\right]N_I d\Gamma = 0$$

The resulting finite element discretization in space of the three above governing equations yields the following system of semi-discrete coupled equations (refer to the appendix for definition of matrices):

$$K\frac{d\bar{u}}{dt} + L\frac{dp}{dt} + UC\frac{d\bar{C}}{dt} = \frac{df}{dt} \quad (34)$$

$$L^T\frac{d\bar{u}}{dt} + S\frac{d\bar{P}}{dt} + HP = F \quad (35)$$

$$CP\frac{d\bar{P}}{dt} + KT\frac{d\bar{C}}{dt} + KD_C + KV_C + PC = F_3 \quad (36)$$

The matrices listed above form the coefficient matrices of the combined equations: -

$$\begin{bmatrix} K & L & UC \\ L^T & S & 0 \\ 0 & CP & KT \end{bmatrix}\frac{d}{dt}\begin{Bmatrix}\bar{u}\\ \bar{P}\\ \bar{C}\end{Bmatrix} + \begin{bmatrix} 0 & L & 0 \\ 0 & H & 0 \\ 0 & CP & KD+KV \end{bmatrix}\begin{Bmatrix}\bar{u}\\ \bar{P}\\ \bar{C}\end{Bmatrix} = \begin{Bmatrix}\frac{df}{dt}\\ F\\ F_3\end{Bmatrix} \quad (37)$$

SOLUTION PROCEDURE

The analysis of displacement, fluid flow and dispersion through a deforming porous medium represents a three-degree of freedom field problem. The most obvious solution procedure of the three-coupled semi-discrete Equations (37) developed in the previous section consists of adding the concentration (C) as an additional variable to the existing nodal variable displacement (u) and pressure (P) parameters to solve the system of equations simultaneously. This is usually done by the *monolithic augmentation approach* (Park and Felippa, 1983), first proposed by Lewis and Karahanglu in 1981, but instead of the *advection-dispersion* equation, the *heat flow equation* was used. Following this approach, Equation (37) becomes:

$$\begin{bmatrix} K & L & UC \\ L^T & S+\bar{\alpha}H\Delta_{tk} & 0 \\ 0 & CP+\alpha PC\Delta_{tk} & kT+\alpha(KD+KV)\Delta_{tk} \end{bmatrix}_{k,\bar{\alpha}}\begin{Bmatrix}\bar{u}\\ \bar{P}\\ \bar{C}\end{Bmatrix}_{tk+\Delta_{tk}} =$$

$$\begin{bmatrix} K & L & UC \\ L^T & S-(1-\bar{\alpha})H\Delta tk & 0 \\ 0 & CP-(1-\bar{\alpha})PC\Delta tk & kT-(1-\bar{\alpha})(KD+KV)\Delta tk \end{bmatrix}_{k,\bar{\alpha}}\begin{Bmatrix}\bar{u}\\ \bar{P}\\ \bar{C}\end{Bmatrix}_{tk} + \begin{Bmatrix}\frac{\partial f}{\partial t}\\ \bar{F}\\ \bar{F}_3\end{Bmatrix}_{k,\bar{\alpha}}\Delta tk \quad (38)$$

The matrices in the above equation need to be evaluated once per time step.

THE HYPERBOLIC MODEL STRESS-STRAIN LAW

Setting out constitutive relations relevant for gypseous soils is still a topic under research. The adoption of the hyperbolic model is an appropriate first step towards this goal, for it is basically a curve fitting technique of available stress-strain curves from laboratory tested specimens (Majeed, 2000). The hyperbolic stress-strain relationship was first proposed by Kondner (1963), and developed by Duncan and Chang (1970), in an attempt to provide a simple framework encompassing the most important characteristics of soil stress-strain behavior, using the data available from conventional laboratory tests such as the unconsolidated un-drained UU tri-axial compression test or the consolidated drained CD tri-axial compression test. The relationship between stress and strain is assumed to be governed by the generalized Hooke's law of elastic deformations which, for plane strain conditions, may be expressed as follows (Wong an Duncan, 1974):

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\tau_{xy} \\ \Delta\sigma_z \end{Bmatrix} = \frac{E_t}{(1+\upsilon_t)(1-\upsilon_t)} \begin{bmatrix} (1-\upsilon_t) & \upsilon_t & 0 & \upsilon_t \\ \upsilon_t & (1-\upsilon_t) & 0 & \upsilon_t \\ 0 & 0 & \left(\frac{1-2\upsilon_t}{2}\right) & 0 \\ \upsilon_t & \upsilon_t & 0 & 1-\upsilon_t \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_y \\ \Delta\gamma_{xy} \\ \Delta\varepsilon_z \end{Bmatrix} \quad (39)$$

Kondner (1963), and Kondner and Zelasko (1963) have shown that the stress-strain curves for a number of soils, both clay and sand, could be approximated reasonably accurate by hyperbolas. This hyperbola can be represented by an equation of the form:

$$(\sigma_1 - \sigma_3) = \frac{\varepsilon}{a + b\varepsilon} \quad (40)$$

$$a = \frac{1}{E_i} \quad (41)$$

$$b = \frac{1}{(\sigma_1 - \sigma_3)_{ult}} \quad (42)$$

It may be noted that a and b are, respectively, the intercept and the slope of the best fit resulting straight line. The asymptotic stress value (σ_1-σ_3) may be related to the compression strength, (σ_1-σ_3), by means of a factor R_f as follows:

$$(\sigma_1 - \sigma_3)_f = R_f (\sigma_1 - \sigma_3)_{ult} \quad (43)$$

By expressing the parameters a and b in terms of the initial tangent modulus value and the compressive strength, Equation (40) may be rewritten as:

$$(\sigma_1 - \sigma_3) = \frac{\varepsilon}{\left[\frac{1}{E_i} + \frac{\varepsilon R_f}{(\sigma_1 - \sigma_3)_f}\right]} \quad (44)$$

The variation of E_i and σ3 is represented by an equation of the form (Janbu, 1963):

$$E_i = kP_a \left\{\frac{\sigma_3}{P_a}\right\}^n \quad (45)$$

Equation (45) can also be used for unloading-reloading conditions, but replacing E by E_{ur} and k by k_{ur}. For saturated soils under un-drained conditions, there is no volume change and Poisson's ratio v_i is equal to one-half for any value of confining pressure. For most other soils the value of v_i decreases with confining pressure and this variation of v_i with σ_3 may be expressed by the equation:

$$v_i = G - F\log\left\{\frac{\sigma_3}{P_a}\right\} \quad (46)$$

where:

G: is the value of v_i at a confining pressure of one atmosphere.

F: is the reduction in v_i for a tenfold increase in σ_3.
The instantaneous slope of the curve representing the variation of ε_a and ε_r is υ_t. By differentiating the equation ($\varepsilon_r = \frac{1}{2}(\varepsilon_v - \varepsilon_a)$) with respect to ε_r, substituting Equation (46) and eliminating the strain using Equations (40 to 44), the tangent value of Poisson's ratio may be expressed in terms of the stresses as follows (Majeed, 2000): -

$$\upsilon_t = \frac{G - F\log\left\{\frac{\sigma_3}{P_a}\right\}}{\left[1 - \frac{d(\sigma_1 - \sigma_3)}{AkP_a\left\{\frac{\sigma_3}{P_a}\right\}^n}\right]^2} \quad (47)$$

where

$$A = \left[1 - \frac{R_f(\sigma_1 - \sigma_3)(1 - \sin\phi)}{2C\cos\phi + 2\sigma_3\sin\phi}\right]$$

DISPERSION TEST

A simplified form of the mass conservation equation was presented in above for the one-dimensional flow through a homogeneous and isotropic porous medium. In order to apply the analysis presented in the derivation of the advection-dispersion equation, it is necessary to have data obtained from one-dimensional flow experiments (Al-Damluji and Al-Rawi, 2005).
The schematic diagram in Figure (3) shows the details of the modified setup. The porous medium box has outer dimensions of (7x30x30) cm. It is made up of 6 mm thick glass sheets and 4mm thick steel sheets where the removable upper cover was manufactured from 1mm thick steel sheets having (3) pores for the outlet water. A matrix of conductivity probes was embedded in the box at various locations to monitor the movement of the gypsum by using an Ohmmeter probe.
Longitudinal dispersion coefficients are determined according to the following equation:

$$D_{11} = 0.5\left\{\frac{t_{84} - t_{16}}{2t_{50}}\right\}v \quad (48)$$

A plot is made between time and relative concentration $(C-C_o)/(C_{max}-C_o)$. The longitudinal apparent dispersivity (A_{11}) is calculated by plotting the values of (D_{11}) against the corresponding values of (v) on log-log paper, as shown in figure (4a). The best fit equation is:

$$D_{11} = 0.3997\,(v)^{0.904} \quad (49)$$

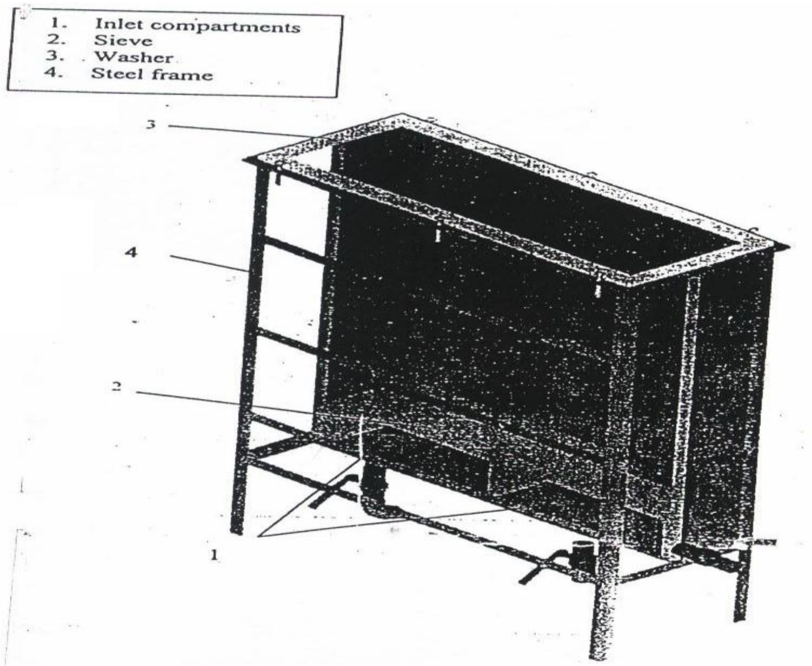


Fig (3) Schematic diagram of the porous medium box

Lateral dispersion coefficients are obtained for various seepage velocities by using the following equation:

$$D_{22} = \frac{V}{L}\left\{\frac{X_{90} - X_{10}}{3.625}\right\}^2 \quad (51)$$

where;

L = distance from inlet, and

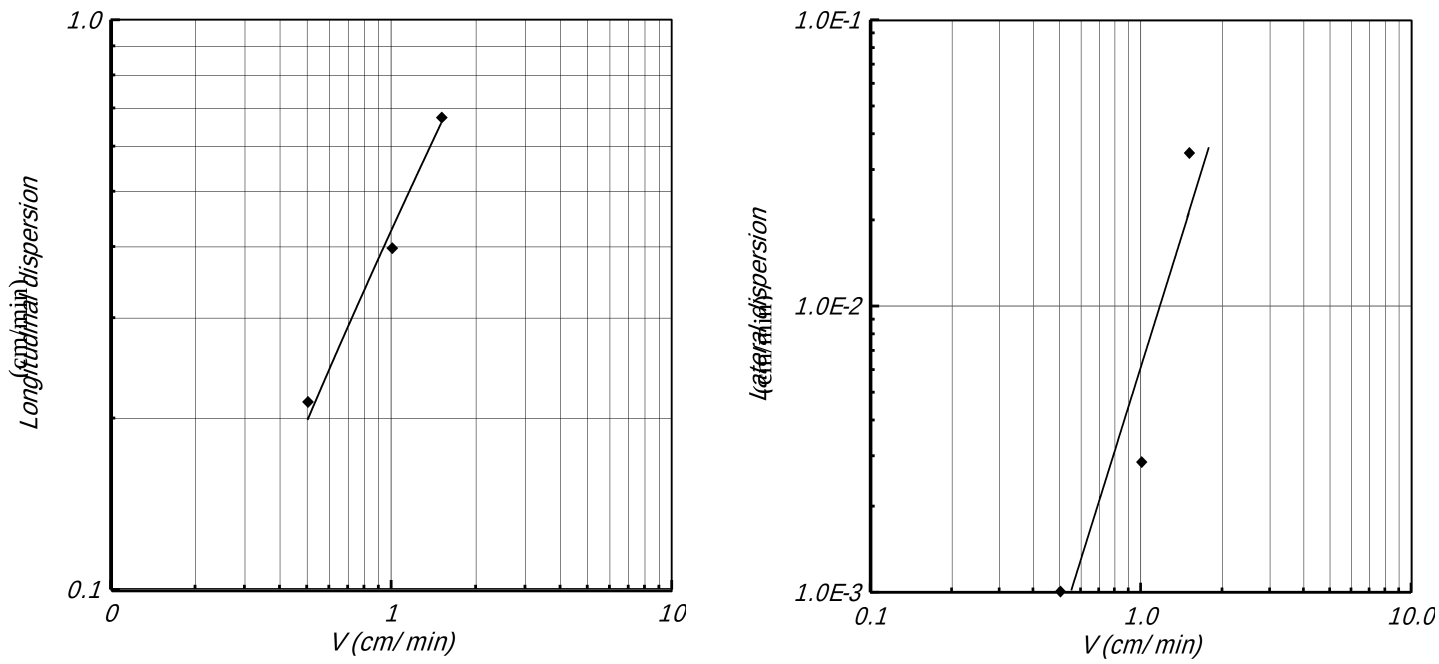
X = lateral distance from 50 % composition point.

The lateral apparent dispersivity (A_{12}) is calculated by plotting the values of (D_{22}) against the corresponding values of (v) on log-log paper, as shown in Figure (4b). The best fit equation is:

$$D_{22} = 0.00255\,(v)^{6.304} \quad (52)$$

Table (1) Coefficients of longitudinal and lateral dispersions (D_{11} and D_{22})

Velocity (cm/min)		1.51	1.006	0.5033
D_{11} (cm/min)	X=5 cm	0.566	0.403	0.244
	X=15 cm	0.755	0.289	0.1617
	X=25 cm	0.705	0.503	0.236
	Average	0.675	0.398	0.2138
D_{22} (cm/min)		0.0344	0.00225	0.0002



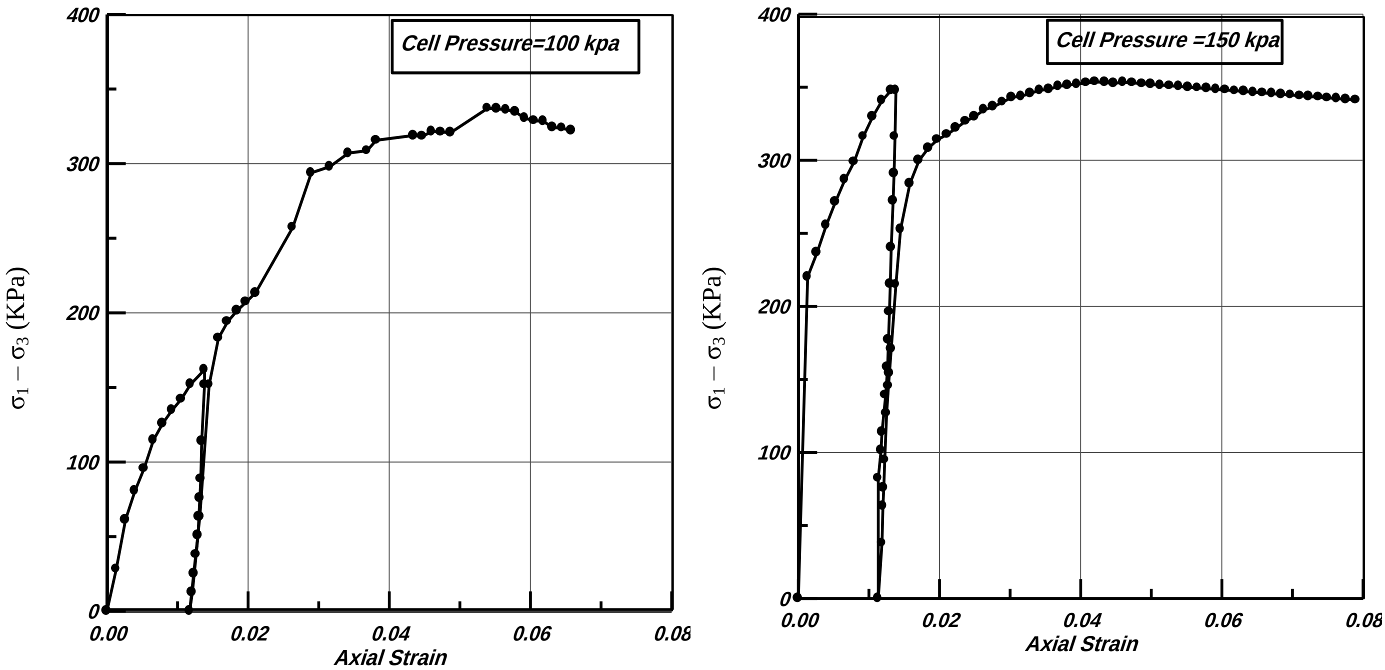
-a- -b-

Fig (4) Longitudinal and lateral dispersive curves for tested soil

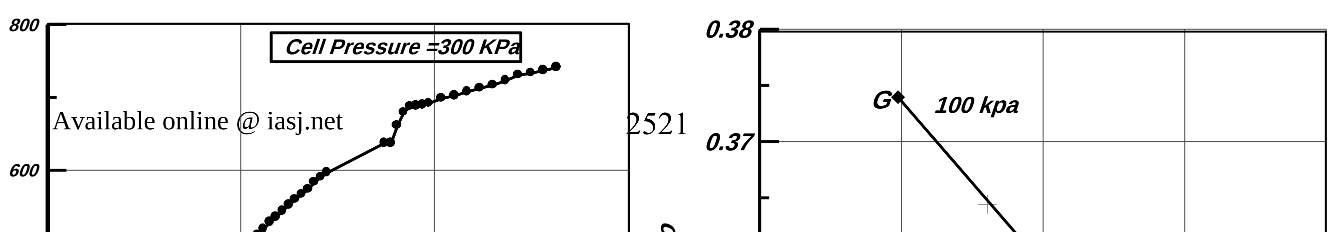
pressures, namely 100, 150 and 300kN/m^2. It is worthy to mention that all these tests were

CONSOLIDATED DRAINED TRIAXIAL TEST (CD TEST)

The consolidated drained tri-axial (CD) test was carried out under three different confining conducted in three different stages which are the *saturation*, the *consolidation* and finally the *shearing* stages. Figures (5a, b, c, and d) show the results of the tests.



-a- -b-



$\sigma_1 - \sigma_3$ (KPa)

Log σ_3/Pa

-c- -d-

Fig (5) Results of drained tri-axial compression (CD) tests under confining stresses of (a)100, (b)150, (c)300kPa, respectively and (d) is the variation of the initial tangent Poisson's ratio with confining stress

Table (2) shows the obtained parameters for the hyperbolic stress-strain relationship. These parameters are used as input data in the developed algorithm. The parameters k, k_{ur} and n from Equation (45), R_f from Equation (43) and G, F and d from Equations (46) and (47) are assumed constant during the dissolution process of gypsum.

Table (2) Parameters of hyperbolic stress-strain relationship

k	k_{ur}	n	Φ	R_f	G	F	D
250	420	3.331	38º.67	0.91	0.374	-0.08	0.0

One Dimensional Problem

The finite element mesh is shown in Figure (6). The width of the mesh is assumed to be equal to 0.5m. An external surface load of 50kN/m2 is assumed to be applied. The time stepping scheme and the material properties for this problem are given in Tables (3) and (4), respectively. There are two cases for this problem with some assumptions:

Case (one): Dissolution of gypsum does not happen in the first problem by assuming $C_o=C_s$. In this case, the developed algorithm solves two equations only, namely, *flow and equilibrium.*

Case (two): Dissolution of gypsum happens by assuming $C_o<C_s$. In this case, the *advection-dispersion* equation is included with the two mentioned in case one equations. Here, C_o will be taken equal to 0.4 gm/l.

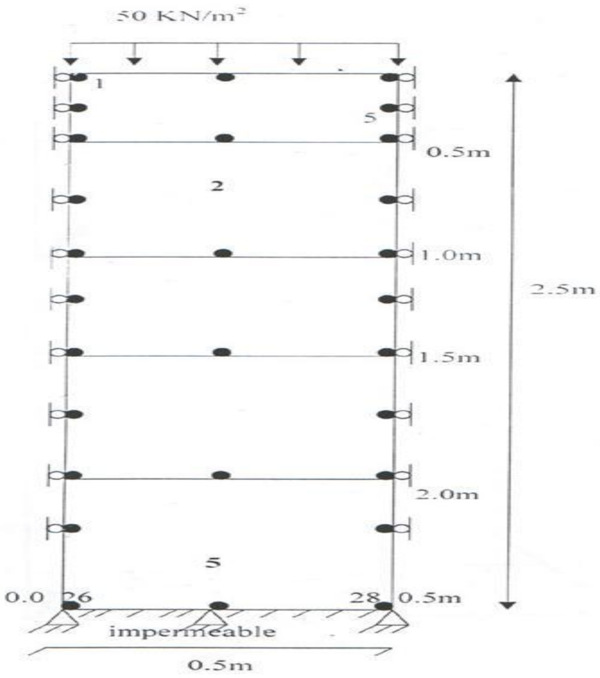


Fig (6) Finite element mesh for the one-dimensional problem

Table (3) Time steps taken for different time values

Time Interval (day)	Number of Time Steps
0.25	8
0.5	8
1.0	104

Figures (7), (8) and (9) show the variation of the elastic modulus, pore water pressure and settlement ratio, respectively at the center of element (1) for cases (one) and (two).

Table (4) Adopted material properties for the one-dimensional problem

The Material properties	Value
Horizontal Coefficient of Permeability, k_h (from tests)	2.5 $*10^{-5}$ m/second
Vertical Coefficient of Permeability, k_v (from tests)	2.5 $*10^{-5}$ m/second
Modulus number k (from tests)	250
Modulus number k_{ur} (from tests)	420
Modulus Exponent n (from tests)	3.331
Angle of internal friction Φ (from tests)	38.67°

Failure ratio R_f (from tests)	0.91
Poisson's ratio Parameter G (from tests)	0.374
Poisson's ratio Parameter F(from tests)	-0.08
Poisson's ratio Parameter d (from tests)	0.0
Elastic modulus (initial) (from tests)	23230 KN/m^2
Poisson's ratio (initial) (from tests)	0.37
Surface area of gypsum (Al-Mufty, 1997)	$(3\alpha\, \chi\, \gamma d)/(nr\, Gs\, \gamma w)$
dissolution rate Ḱ (Al-Mufty, 1997)	$(0.15 + 8.6\, v)/10^5$
Longitudinal Dispersion (D_{11}) (from tests)	$0.3997\, (v)^{0.904}$
Lateral Dispersion (D_{22}) (from tests)	$0.00255\, (v)^{6.304}$
Soil density	18.8 KN/m^2
Cohesion (c) (from tests)	0.0

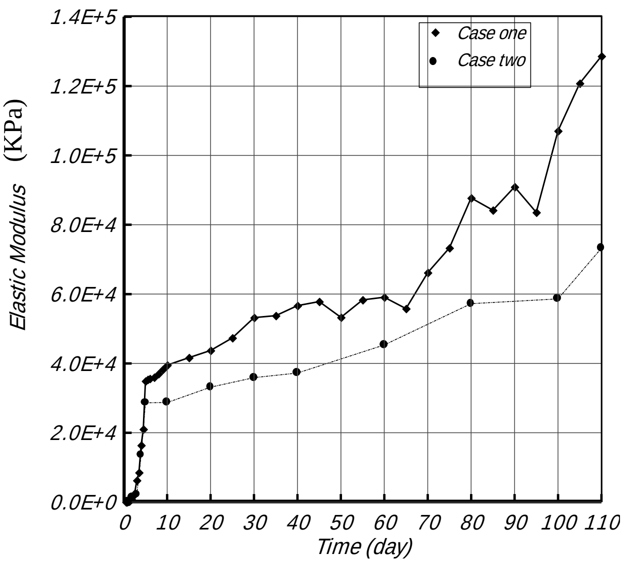


Fig (7) Elasticity modulus-time relation

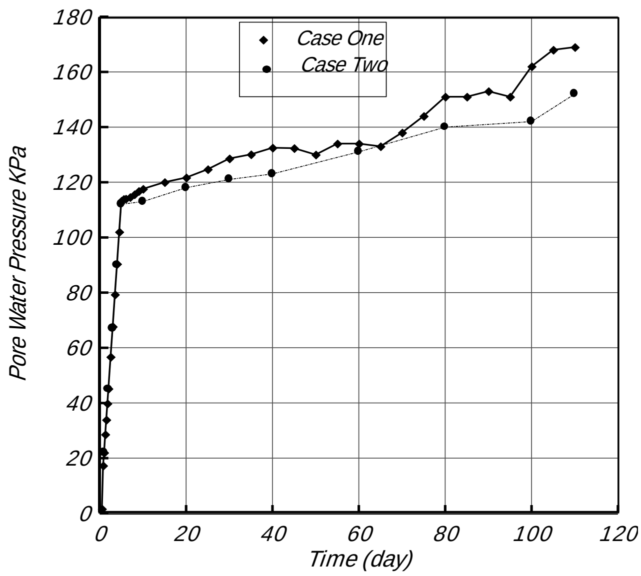


Fig (8) P.W.P.-time relation

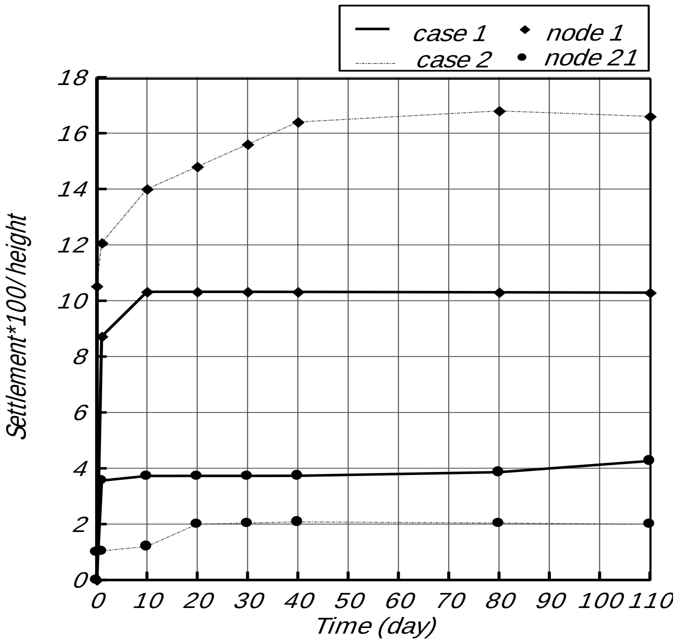


Fig (9) Settlement ratio-time relation

Two Dimensional Problem

Figure (10) shows the finite element mesh. The width of the loaded area B is equal to 9.15m. The applied load is 50kN/m^2.The time stepping scheme and the material properties are the same as those in Tables (3) and (4). Figures (11) and (12) show the results of analysis against time for the elastic modulus and bulk modulus at the center of elements 2, 8, 14, 20, and 26. These results show the variation of values against depth. All the parameters become constant after (5) days of loading and these values are very low when compared to the results of problem one. The values of the elastic modulus are between 500 and 5300 kN/m^2, bulk modulus between 5300 and 22000kN/m^2 and Poisson's ratio between 0.27 and 0.365. Figures (13) and (14) show the variation of pore-water pressure against depth and the settlement against time for cases one and two, respectively.

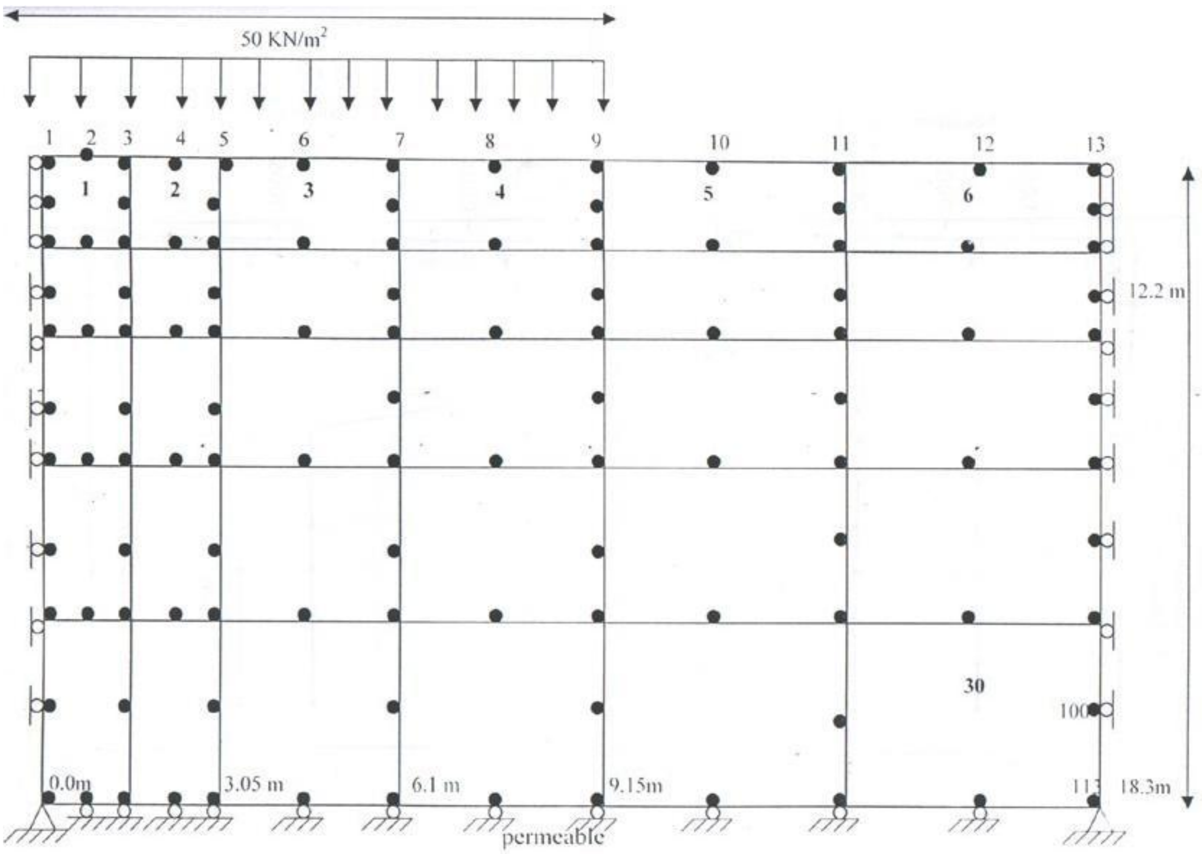
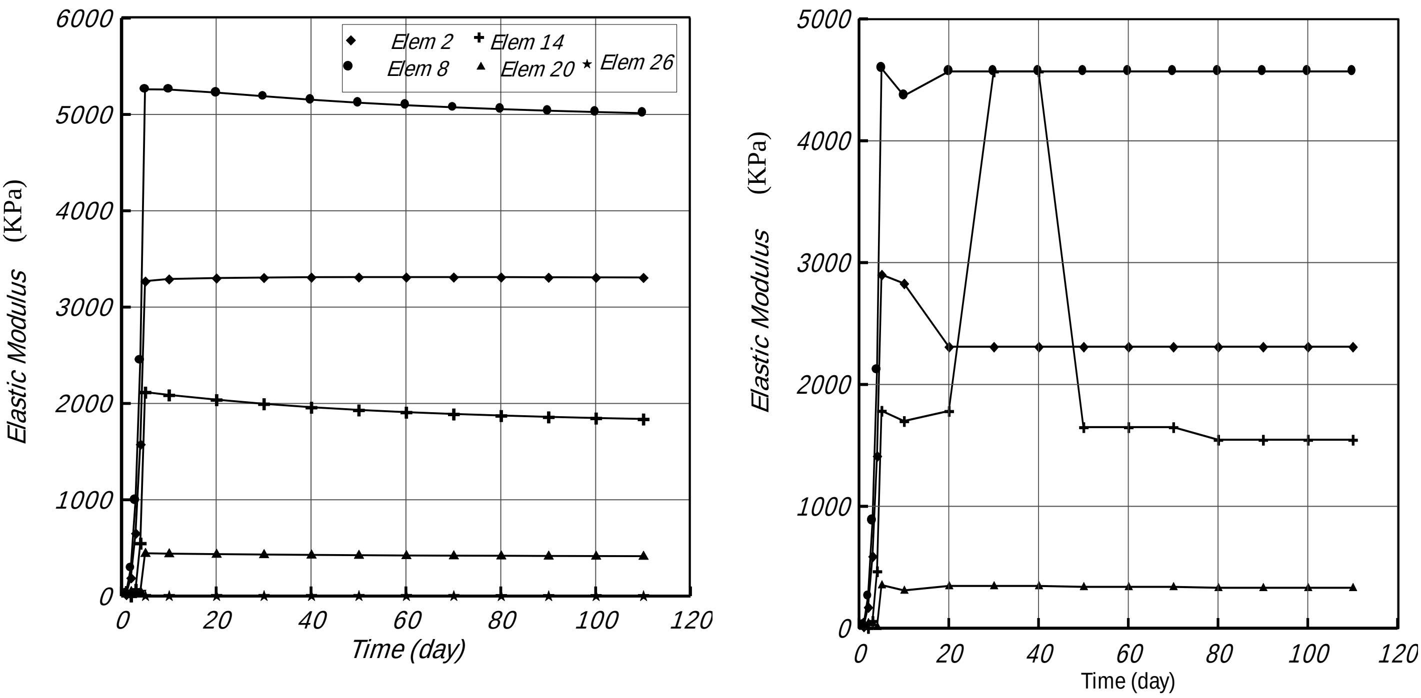


Fig (10) Finite element mesh for the two-dimensional problem



-a- -b-

Fig (11) Variation of elastic modulus with time for (a) case one; and (b) case two

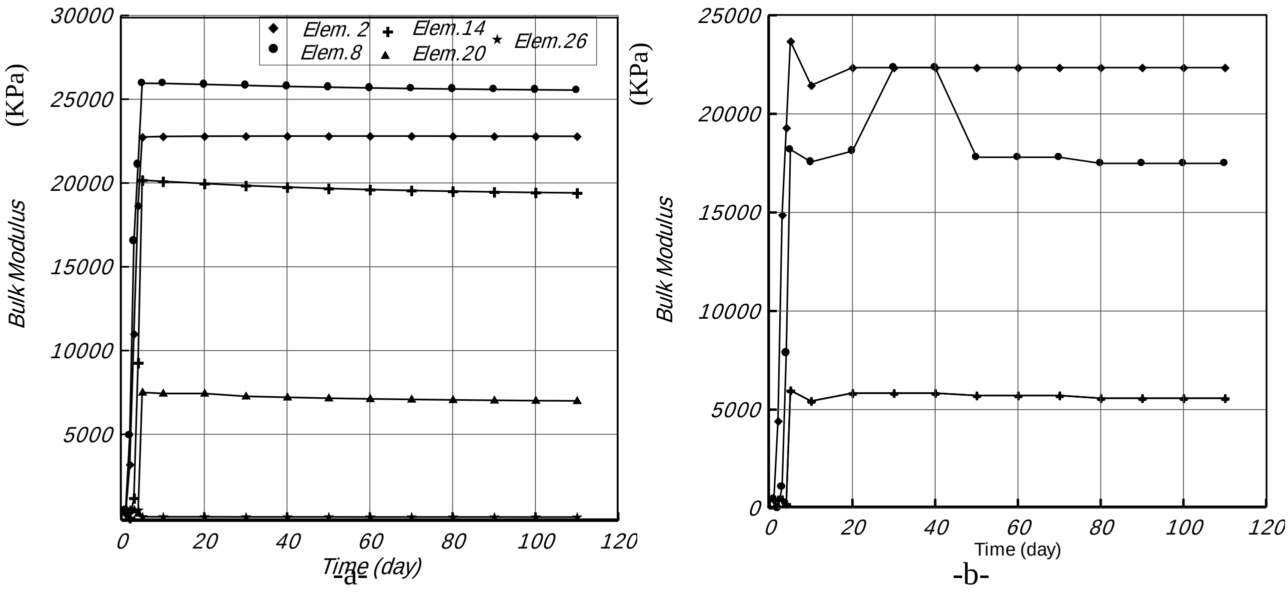


Fig (12) Variation of bulk modulus with time for (a) case one; and (b) case two

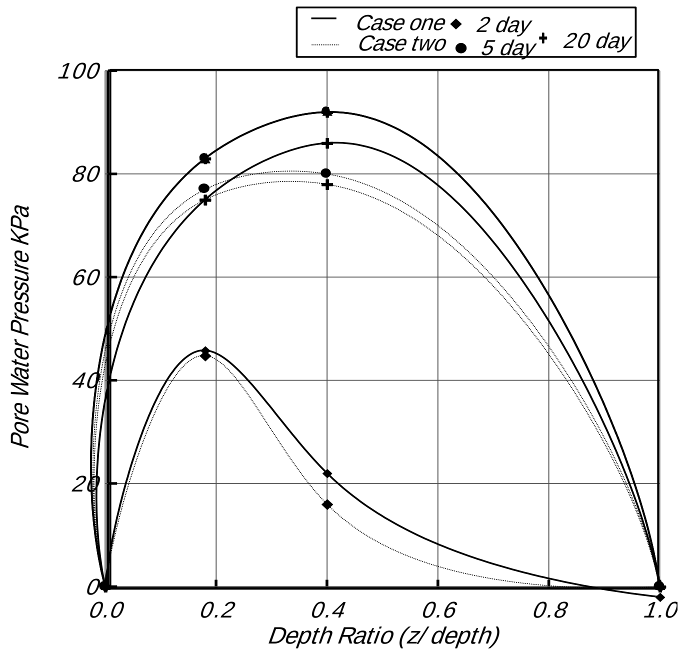


Fig (13) Pore water pressure-depth relation at center of elements 2, 8, 14, 20 and 26

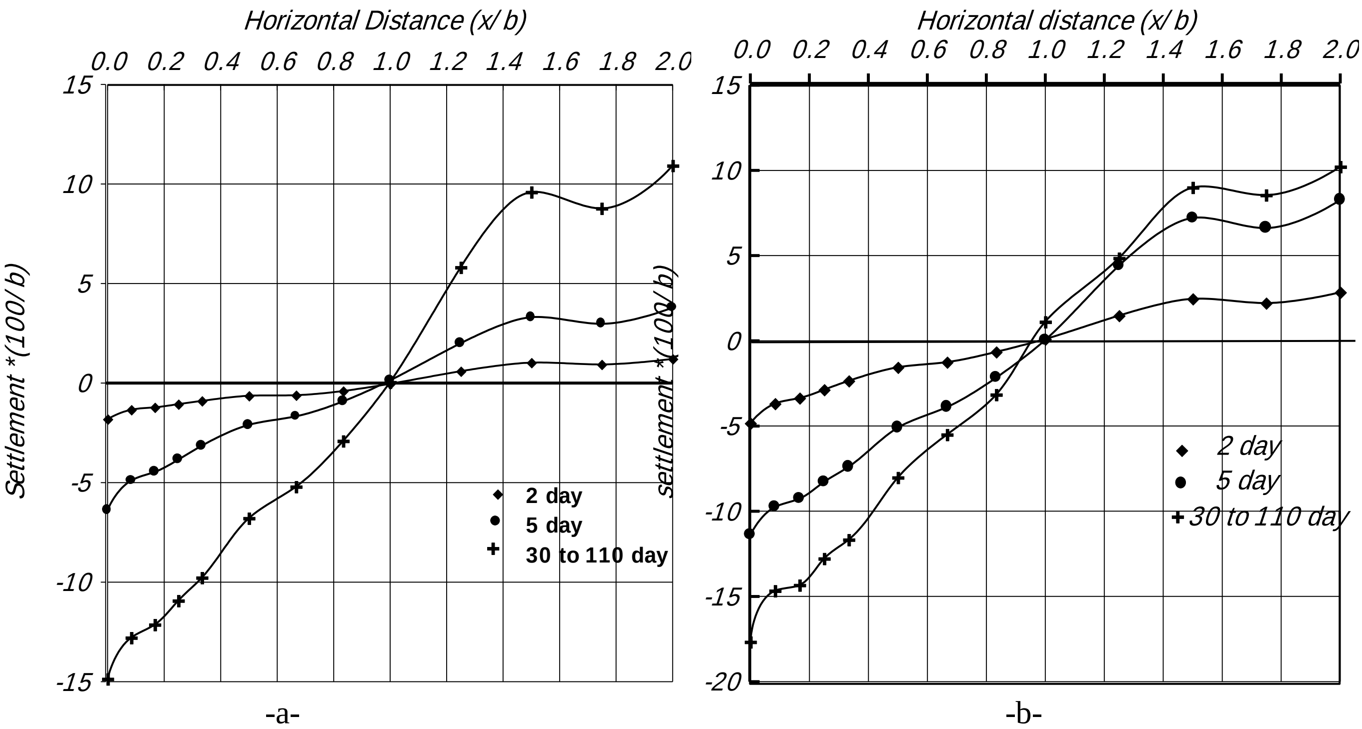


Fig (14) Variation of settlement with time in nodes 1 to 13; (a) case one and (b) case two

CONCLUSIONS

* Results of analysis by using the finite element method show a high effect of dissolution of gypsum on the values of elastic modulus and Poisson's ratio. These parameters and pore water pressures decrease with the increase in dissolution. Dissolution of gypsum is the main reason for this decrease. In addition, there is a high difference in values between the one-dimensional and two-dimensional cases.
* Assuming constant values of hyperbolic stress-strain relationship parameters during the dissolution of gypsum is not correct, because when the content of gypsum varies, some parameters undergo change like specific gravity, density, cohesion and the angle of internal friction. This implies that the values for hyperbolic parameters must vary as the process of dissolution progresses.
* The results of analysis show the soil in a dense state, because the value of the elastic modulus is more than 3000 kN/m^2 and Poisson's ratio more than 0.3. This is similar to laboratory test results.
* The one-dimensional and two-dimensional problems have shown that the dissolution of gypsum decreases with the increase of depth.
* The area of a gypsum particle depends mainly on the diameter. This study assumes that all particles have the same diameter. If the particles considered have varying diameters, then this would have its effect on the results.

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Abbreviations:

ASCE = American Society of Civil Engineers.

PWP = pore water pressure.

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List of Symbols:

A : is area of section of flow (m^2).

A_{11} : is the longitudinal apparent dispersivity.
A_{12} : is the lateral apparent dispersivity.
a : is the intercept of the best fit resulting straight line.
b : is the slope of the best fit resulting straight line.
b : are the body forces.
B : is the bulk modulus.
C : is the concentration of gypsum in water (mg/l).
$\hat{C}$: is the concentration of gypsum in the fluid element.
Co : is the original value of concentration.
c : is the cohesion (from tests).
DT : is the tangent matrix.
Dd : is the coefficient of molecular diffusion.
D_{11} : is the longitudinal dispersion **(**D_{11}) (from tests).
D_{22} : is the lateral dispersion **(**D_{22}) (from tests)
dp : is the pore water pressure.
d : is Poisson's ratio Parameter (from tests)
F : is the reduction in Poisson's ratio v_i for a tenfold increase in σ_3 (from tests).
G : is the value of Poisson's ratio v_i at a confining pressure of one atmosphere.
G_g : is the specific gravity of gypsum (usually taken as 2.32).

g : is the gravitational acceleration.

h : is the total head.
Ks : is the bulk modulus of the solid phase.
k : is the absolute permeability matrix of the medium,
k_h : is the horizontal coefficient of permeability, (from tests).
k_v : is the vertical coefficient of permeability, (from tests).
K : is the modulus number (from tests).
k_{ur} : is the modulus number (from tests).
K : is the dissolution rate Ḱ.
L : is the distance from inlet.
M : is the mass.
M_{VE} : is the mass contained inside the volume element.
MP : is the mass source or sink term which is positive when a source and negative when a sink.
m : is equal to unity for the normal stress components and zero for the shear stress components.
n : is the porosity.
n : is the modulus exponent (from tests).
O : is the original value of the variable.
P : is the fluid pressure.
Pa : is the atmospheric pressure.
Po : is the original value of pressure.
R_f : is the failure ratio (from tests).
t : is time (sec).

$\hat{t}$: is the boundary traction ($\hat{t}$) specified at the boundary.

u : is the unknown displacement vector.

υ : is Poisson's ratio.

v_i : is the velocity (m/sec).

V_w : is the volume of water (L^3).

V_o : is the initial volume of element (L^3).

$\overrightarrow{\hat{V}_t}$: is the velocity of the gypsum in the fluid element with respect to a fixed coordinate system.
$\overrightarrow{\hat{V}}$: is the volumetric velocity of the fluid element.

x : is the lateral distance from 50 % composition point.
α : is the proportionality factor relating C and ρ.
β : is the fluid compressibility.
$d\varepsilon$: is the total strain of skeleton.
$d\varepsilon p$: is the overall volumetric strain caused by the uniform compression of the particles due to the pressure of the pore fluid.
$d\varepsilon_m$: the strain due to the dissolved mass of calcium sulphate,

$d\varepsilon_o$: represents all other strains not directly associated with stress changes, e.g. swelling, thermal, …etc.

Φ : is the angle of internal friction (from tests).

Γ : is the boundary.

γ_w : is the density of water (kg/m3).

μ : is the dynamic viscosity of the fluid.

ρ : is the density of the fluid.

ρ_o : is the original value of density.

σ : is the total stress.

Ω : is the domain.

Appendix:

$$K = -\int_\Omega B^T DB d\Omega$$

$$L = \int_\Omega B^T N d\Omega$$

$$UC = \int_\Omega B^T D(V_N A_N - V_Y A_Y)\left(\frac{\Delta t}{G_g \gamma_w V_o}\right) N d\Omega$$

$$df = -\int_\Omega N^T dbd\Omega - \int_\Gamma N^T dt' d\Gamma - \int_\Omega B^T D d\varepsilon_o d\Omega$$

$$L^T = \int_\Omega N^T m^T B d\Omega$$

$$S = -\int_\Omega N^T \left(\frac{n}{K_w}\right) N d\Omega$$

$$H = \int_\Omega (\nabla N)^T \left(\frac{k}{\mu}\right) \nabla \ddot{N} d\Omega$$

$$F = -\int_\Gamma N^T q d\Gamma - \int_\Omega (\nabla N)^T \left(\frac{k}{\mu}\right) pghd\Omega$$

$$CP = \iint C N_I N_J dxdy$$

$$KT = \iint N_I N_J dxdy$$

$$KD = \iint \left\{ Dxx\left(\frac{\partial N_I}{\partial x}\right)\left(\frac{\partial N_J}{\partial x}\right) + Dyy\left(\frac{\partial N_I}{\partial y}\right)\left(\frac{\partial N_J}{\partial y}\right) \right\} dxdy$$

$$KV = \iint \left\{ VxN_I\left(\frac{\partial N_J}{\partial x}\right) + VyN_I\left(\frac{\partial N_J}{\partial y}\right) \right\} dxdy$$

$$PC = \iint \left\{ CVxN_I\left(\frac{\partial N_J}{\partial x}\right) + CVyN_I\left(\frac{\partial N_J}{\partial y}\right) \right\} dxdy$$

$$F_3 = \iint \left\{ NVx(Cp - C)\left(\frac{A_x}{nV_o}\right) + NVy(Cp - C)\left(\frac{A_y}{nV_o}\right) \right\} dxdy$$