DERIVATION OF STIFFNESS MATRIX FOR A GENERAL TWO DIMENSIONAL CURVED ELEMENT IN GENERAL GLOBAL COORDINATES SYSTEM

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ABSTRACT

In the present paper , the derivation of stiffness matrix for a general two dimensional curved element in global coordinates system is presented. The derivation depends on the assumption that any curved in-plane element can be approximated by a specified curve in polar coordinates. The polar curve assumed in this polar depends on some variables that enable it to represent any two dimensional curved element. The derivation process accompanied by complex integrals which are evaluated by using (**Gaussian Quadrature**) method of numerical integration. One numerical example is presented to verify the accuracy and efficiency of the derived stiffness matrix. The verification contains a comparison with the results of the exact solution. Very good agreement is obtained between the results of the derived stiffness matrix and the results of the exact model.

الخلاصة

في هذا البحث تم اشتقاق مصفوفة الجساءة لأي عنصر مقوس ثنائي الأبعاد وبنظام الإحداثيات العام. الاشتقاق يعتمد على فرضية أن أي عنصر مقوس ثنائي الأبعاد يمكن أن يقرب بمنحني معين بالإحداثيات القطبية. المنحني القطبي المفروض في هذا البحث يعتمد على بعض المتغيرات التي تمكنه من تمثيل أي عنصر منحني ثنائي الأبعاد. عملية الاشتقاق صوحبت بتكاملات معقدة والتي تم حسابها باستخدام طريقة (**Gaussian Quadrature**) للتكاملات العددية. تم حل مثال واحد لغرض إثبات دقة وفاعلية المصفوفة الجديدة. تم مقارنة النتائج التي تم الحصول عليها مع الحل المضبوط ومن خلال المقارنة تم الحصول على نتائج قريبة جدا من الحل المضبوط وبفرق لايزيد عن (%1,7).

KEYWORDS

Stiffness matrix; two dimensional element; general curved element; global coordinates

INTRODUCTION

In the past, the curve beam or arch represents one of the few structural systems which make it possible to cover lar_e spans. The earliest inhabitants developed the arch as an important element of their architecture as expressed by remaining bridges, aqueducts and large public buildings. To day, the same importance is presented especially in bridge construction. Typical forms of curved beams are: circular arches, cantilever-curved beams, elliptical arches, cubic, square arches and catenary curved beams. Most of early modern curved beams have semicircular shapes. Now, curved beams or arched structures are constructed in different shapes and from variable materials as brick, steel, reinforced concrete, ferrocement and timber. The main aim of the curved beam is to enhance the load carrying capacity, which may come from stiffening beavior due to membrane action. A literature survey indicates that a substantial amount of works that deal with the analysis of arches by using circular curved finite element. **Just(1982)** presented the exact $(6*6)$ stiffness matrix for a circularly curved beam subjected to loading in its own plane. This matrix was derived from the governing differential equations and from the finite element procedure. The strain energy contributed axial and flexural actions were considered. **Akhtar(1987)** expressed the stiffness matrix of a single circular member of uniform cross-section. He also obtained the fixed end actions due to concentrated load acting at any point on the member making any angle with radial direction at this point. The effect of shear deformations was neglected. **Litewka and Rakowski(1998)** derived the exact stiffness matrix for a curved beam element with constant curvature(circular curved element). The plane two node six-degree-of-freedom element was considered. **Hadi(2002)** developed a circular curved beam element stiffness matrix. He included the effect of shear deformations. The derived matrix is used in the nonlinear analysis of reinforced concrete circular arches. It is clear from the preceding review that there is no formulation of a more general stiffness matrix for a curved element including circular and non-circular curved elements. The objective of the present paper is to derive a more general stiffness matrix that deals with circular and non-circular curved elements by using the principle of the strain energy. In addition, the derived stiffness matrix is in global coordinates system and can be applied on any curved element without any transformation.

- Derivation of Stiffness Matrix For a General Curved Beam Element:

Before the derivation of stiffness matrix, a general equation for any curved in-plane element must be found. In this paper, a general equation in polar coordinates is suggested to represent any plane curved element. The suggested equation is:

$$r = a \cos n\theta \qquad (1)$$

Which represents a family of (flower-shaped) curves or roses depending on the value of (n) equally spaced petals of radius $(a \ at \theta = 0)$. So, if one takes the first quadrant of eq.(1) , any plane curved element can be fitted by choosing a specified values for (n) and (a) depends on span length and a number of known $(x-y)$ coordinates for the curve. For $(n=1)$, the curve represents a circle as shown in **Fig.1** and for $(n=2)$ represents a rose of $(2n)$ equally spaced petals if (n) is even $(n=2,4,6,....)$ and of (n) equally spaced petals if (n) is odd $(n=3,5,7,.....)$. As (n) reaches high values, the above eq.(1) represent a straight beam. So, the chosen polar curve represents an infinite number of curves varies from a circle $(n=1)$ and the degree of curvature will decrease with the increment of (n) until it will reach to a straight element $(n \approx \infty)$ or large values. The above explanation can be seen in **Fig.1**.

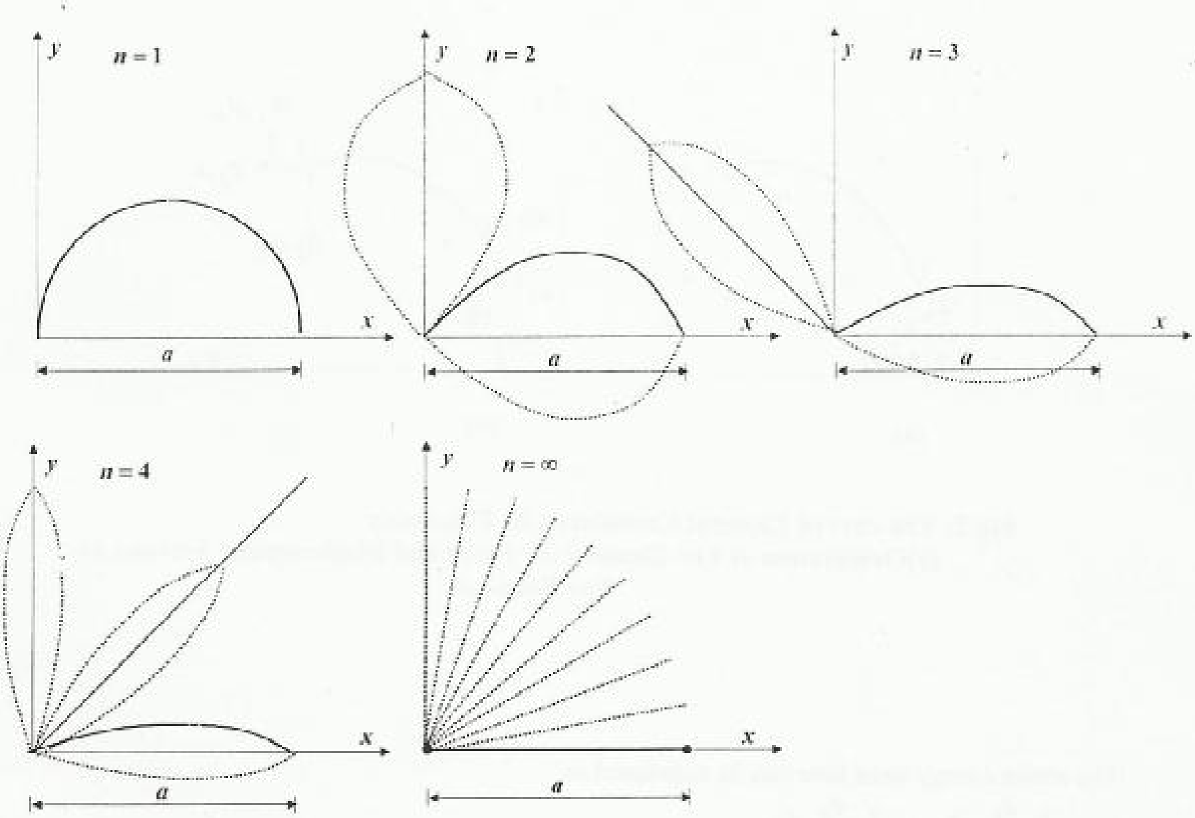

Fig.1: The Graph of $r = a \cos n\theta$

The derivation of the stiffness matrix depends on the principle of the strain energy. The model forces and displacements of a curved element are shown in Fig.2. The internal forces can be expressed in terms of the nodal forces at node (i) by using the static equilibrium equations. The global coordinates system is considered for the directions of the nodal forces and displacements. Hence, by using the suggested polar equation, the following internal forces can be obtained

$$P = P_i \cos\beta + Q_i \sin\beta \tag{2}$$

Where (β) is the inclination angle of the tangent to the polar curve at a point having coordinates (r,θ) which can be expressed as

$$\tan\beta = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{n \sin n\theta \sin\theta - \cos n\theta \cos\theta}{n \sin n\theta \cos\theta + \cos n\theta \sin\theta} \tag{3}$$

In which $x = r\cos\theta,\ y = r\sin\theta,\ r = a\cos n\theta$

$$M = Q_i(r\cos\theta - r_2\cos\theta_2) - (P_i(r\sin\theta - r_2\sin\theta_2) + M_i) \tag{4}$$

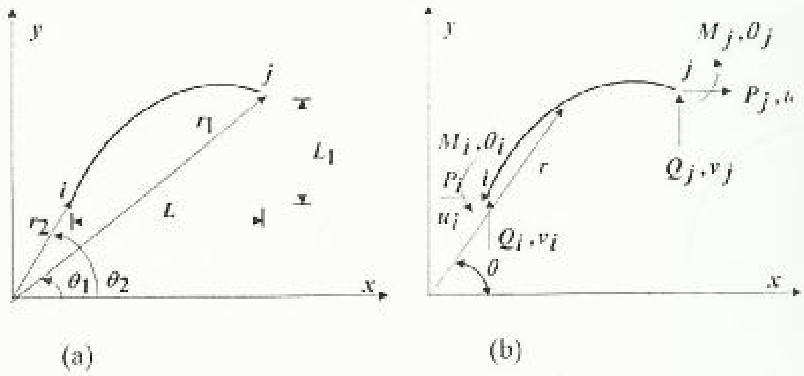

Fig.2: The curved Element Considered In This Study
(a)Orientation of The Element (b) Force and Displacement Systems on The Element

The strain energy used here can be expressed as

$$U = \frac{1}{2EI}\int_{\theta_1}^{\theta_2} M^2 ds + \frac{1}{2AE}\int_{\theta_1}^{\theta_2} P^2 ds \tag{5}$$

Where $\left(ds = a\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\; d\theta\right)$ is the length of the curve segment in polar coordinates used to find the strain energy along the curve , **Anton et al (2002).**
By substituting expressions (2) and (4) into the equation of the strain energy (5) one can get

$$U = \frac{1}{2EI}\begin{bmatrix} Q_i^2(a_1) - 2Q_iP_i(a_2) - 2Q_iM_i(a_3) + P_i^2(a_4) + 2P_iM_i(a_5) + M_i^2(a_6) + \\ \frac{I}{A}(P_i^2(a_7) + P_iQ_i(a_8) + Q_i^2(a_9)) \end{bmatrix} \tag{6}$$

Where

$$a_1 = \int_{\theta_1}^{\theta_2}\left[\left[a^3(\cos n\theta)^2(\sin n\theta)^2 - 2a^2r_2\cos\theta_2(\cos n\theta\cos\theta) + ar_2^2(\cos\theta_2)^2\right]\left(\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right)\right]d\theta \tag{7}$$

$$a_2 = \int_{\theta_1}^{\theta_2}\left[\left[\frac{1}{2}a^3(\cos n\theta)^2(\sin 2\theta) - a^2r_2\cos\theta_2(\cos n\theta\sin\theta) - a^2r_2\sin\theta_2(\cos n\theta\cos\theta) + \frac{1}{2}ar_2^2(\sin 2\theta_2)\right]\left(\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right)\right]d\theta$$

(8)

$$a_3 = \int_{\theta_1}^{\theta_2}\left[\left[a^2(\cos n\theta\cos\theta) - ar_2\cos\theta_2\right]\left(\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right)\right]d\theta \tag{9}$$

$$a_4 = \int_{\theta_1}^{\theta_2}\left[\left[a^3(\cos n\theta)^2(\sin\theta)^2 - 2a^2r_2\sin\theta_2(\cos n\theta\sin\theta) + ar_2^2(\sin\theta_2)^2\right]\left(\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right)\right]d\theta \tag{10}$$

$$a_5 = \int_{\theta_1}^{\theta_2}\left[\left[a^2(\cos n\theta \sin\theta) - ar_2 \sin\theta_2\right]\left[\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right]\right]d\theta \tag{11}$$

$$a_6 = \int_{\theta_1}^{\theta_2}\left[a\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right]d\theta \tag{12}$$

$$a_7 = \int_{\theta_1}^{\theta_2}\left[a(\cos\beta)^2\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right]d\theta \tag{13}$$

$$a_8 = \int_{\theta_1}^{\theta_2}\left[a(\sin 2\beta)\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right]d\theta \tag{14}$$

$$a_9 = \int_{\theta_1}^{\theta_2}\left[a(\sin\beta)^2\sqrt{(\cos n\theta)^2 + n^2(\sin n\theta)^2}\right]d\theta \tag{15}$$

Integrations of eqs.(7 to 15) is complicated, so, it can be found by using **Gaussian-Quadrature** method of numerical integration (see Appendix –A). It is found that three Gaussian points give close results to the exact solution for the example presented in this paper. The stiffness coefficients corresponding to the degrees of freedom shown in **Fig.2(b)** can be obtained by using **Castigliano's** second theorem **Boresi and Schmidt (2003),** which states that the deflection caused by an external force is equal to the partial derivative of the strain energy (U) with respect to that force.
The partial derivatives of the strain energy (U) with respect to (P_i), (Q_i) and (M_i) respectively are:

$$\frac{\partial U}{\partial P_i} = \frac{1}{2EI}\left[P_i(c_1) + Q_i(c_2) + M_i(c_3)\right] \tag{16}$$

$$\frac{\partial U}{\partial Q_i} = \frac{1}{2EI}\left[P_i(c_2) + Q_i(c_4) + M_i(c_5)\right] \tag{17}$$

$$\frac{\partial U}{\partial M_i} = \frac{1}{2EI}\left[P_i(c_3) + Q_i(c_5) + M_i(c_6)\right] \tag{18}$$

Where

$$\left.\begin{aligned} c_1 &= 2(a_4) + \frac{2aI}{A}(a_7) \\ c_2 &= \frac{aI}{A}(a_8) - 2(a_2) \\ c_3 &= 2(a_5) \\ c_4 &= 2(a_1) + \frac{2aI}{A}(a_9) \\ c_5 &= -2(a_3) \\ c_6 &= 2(a_6) \end{aligned}\right\} \tag{19}$$

The stiffness coefficient (K_{ij}) can be defined as the force of type (i) which required to cause a unit displacement of type (j) with all other types of displacements equal to zero. Therefore, equations (16), (17) and (18) will be used to fined the stiffness matrix of the element.

2.1 Axial Stiffness

Consider the element shown in **Fig.3** which is subjected to a unit axial displacement. The stiffness coefficients corresponding to that displacement can be obtained by setting expressions (16),(17) and (18) equal to 1,0 and 0 respectively, hence

$$P_i(c_1) + Q_i(c_2) + M_i(c_3) = 2EI \tag{20}$$

$$P_i(c_2)+Q_i(c_4)+M_i(c_5)=0 \tag{21}$$

$$P_i(c_3)+Q_i(c_5)+M_i(c_6)=0 \tag{22}$$

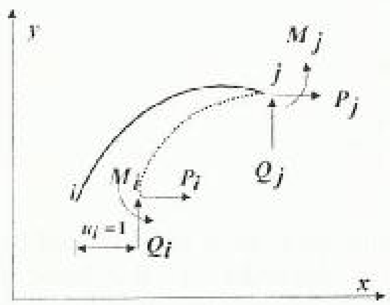

Fig.3: Curved Element Subjected to a Unit Axial Displacement

Solving eqs. (20), (21) and (22) simultaneously yields

$$P_i = \frac{2EI}{A_1} \tag{23}$$

$$Q_i = \frac{2EI}{A_1} Z_1 \tag{24}$$

$$M_i = \frac{-2EI}{A_1}\left[\frac{c_3 + c_5 Z_1}{c_6}\right] \tag{25}$$

Where

$$A_1 = \left[c_1 + c_2 Z_1 - \frac{c_3}{c_6}(c_3 + c_5 Z_1)\right], Z_1 = \frac{(\frac{c_5 c_3}{c_6} - c_2)}{(c_4 - \frac{c_5^2}{c_6})}$$

From equilibrium requirements

$$P_j = -P_i = \frac{-2EI}{A_1} \tag{26}$$

$$Q_j = -Q_i = \frac{-2EI}{A_1} Z_1 \tag{27}$$

$$M_j = Q_i L - (P_i L_1 + M_i)$$

$$= \frac{2EI}{A_1}\left[Z_1 L - (L_1 - (\frac{c_3 + c_5 Z_1}{c_6}))\right] \tag{28}$$

*** Lateral Stiffness**

Proceeding as in the previous section, stiffness coefficients due to a unit lateral displacement at a node (i) (**Fig.4**) can be found by making expressions (16),(17)and (18) equal to 0,1 and 0 respectively, hence

$$P_i(c_1)+Q_i(c_2)+M_i(c_3)=0 \quad (29)$$
$$P_i(c_2)+Q_i(c_4)+M_i(c_5)=2EI \quad (30)$$
$$P_i(c_3)+Q_i(c_5)+M_i(c_6)=0 \quad (31)$$

Again, by solving the above three equations simultaneously, the following expressions can be obtained

$$P_i=\frac{-2EI}{A_2}\left[\frac{c_2+c_3Z_2}{c_1}\right] \quad (32)$$

$$Q_i=\frac{2EI}{A_2} \quad (33)$$

$$M_i=\frac{2EI}{A_2}Z_2 \quad (34)$$

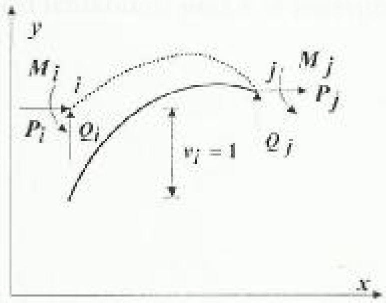

Fig.4: Curved Element Subjected to a Unit Lateral Displacement

Where

$$A_2=\left[c_4+c_5Z_2-\frac{c_2}{c_1}(c_2+c_3Z_2)\right], Z_2=\frac{(\frac{c_3c_2}{c_1}-c_5)}{(c_6-\frac{c_3^2}{c_1})}$$

Also, due to equilibrium requirements

$$P_j=\frac{2EI}{A_2}\left[\frac{c_2+c_3Z_2}{c_1}\right] \quad (35)$$

$$Q_j=\frac{-2EI}{A_2} \quad (36)$$

$$M_j=Q_iL-(P_iL_1+M_i)$$
$$=\frac{2EI}{A_2}\left[L-(Z_2-(\frac{c_2+c_3Z_2}{c_1})L_1)\right] \quad (37)$$

- Rotational Stiffness

Stiffness coefficients corresponding to a unit rotational displacement at a node (i) **(Fig.5)** can be found by setting expressions (16),(17)and (18) equal to 0,0 and 1 respectively, hence

$$P_i(c_1)+Q_i(c_2)+M_i(c_3)=0 \quad (38)$$
$$P_i(c_2)+Q_i(c_4)+M_i(c_5)=0 \quad (39)$$
$$P_i(c_3)+Q_i(c_5)+M_i(c_6)=2EI \quad (40)$$

Solving eqs.(38),(39), and (40) simultaneously, one can get

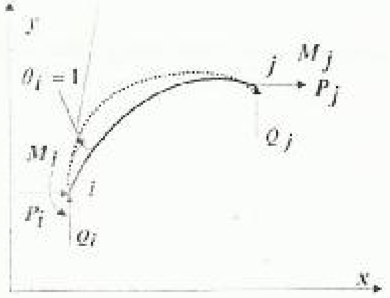

Fig.5: Curved Element Subjected to a Unit Rotational Displacement

$$P_i = \frac{-2EI}{A_3}\left[\frac{c_2 Z_3 + c_3}{c_1}\right] \tag{41}$$

$$Q_i = \frac{2EI}{A_3} Z_3 \tag{42}$$

$$M_i = \frac{2EI}{A_3} \tag{43}$$

In which

$$A_3 = \left[c_6 + c_5 Z_3 - \frac{c_3}{c_1}(c_3 + c_2 Z_3)\right], Z_3 = \frac{(\frac{c_3 c_2}{c_1} - c_5)}{(c_4 - \frac{c_2^2}{c_1})}$$

From equilibrium requirements, the following expressions can be obtained

$$P_j = \frac{2EI}{A_3}\left[\frac{c_2 Z_3 + c_3}{c_1}\right] \tag{44}$$

$$Q_j = \frac{-2EI}{A_3} Z_3 \tag{45}$$

$$M_j = Q_i L - (P_i L_1 + M_i)$$
$$= \frac{2EI}{A_3}\left[Z_3 L - (1 - (\frac{c_2 Z_3 + c_3}{c_1})L_1)\right] \tag{46}$$

All other stiffness coefficients can be found from symmetry and equilibrium requirements. The coefficients of the $(6*6)$ stiffness matrix in the global coordinates system according to the degrees of freedom shown in **Fig.2 (b)** is as follows:

$$K_{11} = \frac{2EI}{A_1} \tag{47}$$

$$K_{21} = \frac{2E^{I}}{A_1} Z_1 \tag{48}$$

$$K_{31} = \frac{-2EI}{A_1}\left[\frac{c_3 + c_5 Z_1}{c_6}\right] \tag{49}$$

$$K_{41} = \frac{-2EI}{A_1} \tag{50}$$

$$K_{51} = \frac{-2EI}{A_1} Z_1 \qquad (51)$$

$$K_{61} = \frac{2EI}{A_1}\left[Z_1 L - (L_1 - (\frac{c_3 + c_5 Z_1}{c_6})) \right] \qquad (52)$$

$$K_{22} = \frac{2EI}{A_2} \qquad (53)$$

$$K_{32} = \frac{2EI}{A_2} Z_2 \qquad (54)$$

$$K_{42} = \frac{2EI}{A_2}\left[\frac{c_2 + c_3 Z_2}{c_1} \right] \qquad (55)$$

$$K_{52} = \frac{-2EI}{A_2} \qquad (56)$$

$$K_{62} = \frac{2EI}{A_2}\left[L - (Z_2 - (\frac{c_2 + c_3 Z_2}{c_1}) L_1) \right] \qquad (57)$$

$$K_{33} = \frac{2EI}{A_3} \qquad (58)$$

$$K_{43} = \frac{2EI}{A_3}\left[\frac{c_2 Z_3 + c_3}{c_1} \right] \qquad (59)$$

$$K_{53} = \frac{-2EI}{A_3} Z_3 \qquad (60)$$

$$K_{63} = \frac{2EI}{A_3}\left[Z_3 L - (1 - (\frac{c_2 Z_3 + c_3}{c_1}) L_1) \right] \qquad (61)$$

$$K_{44} = \frac{2EI}{A_1} \qquad (62)$$

$$K_{54} = \frac{2EI}{A_1} Z_1 \qquad (63)$$

$$K_{64} = \frac{-2EI}{A_1}\left[Z_1 L - (L_1 - (\frac{c_3 + c_5 Z_1}{c_6})) \right] \qquad (64)$$

$$K_{55} = \frac{2EI}{A_2} \qquad (65)$$

$$K_{65} = \frac{-2EI}{A_2}\left[L - (Z_2 - (\frac{c_2 + c_3 Z_2}{c_1}) L_1) \right] \qquad (66)$$

$$K_{66} = \left[(K_{62}) L - ((K_{61}) L_1 + K_{63}) \right] \qquad (67)$$

The above coefficients can be written in a matrix form as follows

$$[K] = \begin{bmatrix} K_{11} & & & & & \\ K_{21} & K_{22} & & sym. & & \\ K_{31} & K_{32} & K_{33} & & & \\ K_{41} & K_{42} & K_{43} & K_{44} & & \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix}$$

- Numerical Example

To verify the validity and efficiency of the derived matrix, a cantilever-curved beam shown in **Fig.6** is analyzed by using the derived stiffness matrix. The curved beam is analyzed previously by **Just(1982)** by using an exact circularly curved beam element derived by him. In this element, the strain energy contributed by axial and flexural actions was considered. Three subtended angles (γ) of the curved cantilever were investigated. The beam was analyzed by using a single curved element and also by approximating it into a various numbers of equal straight segments. In the present study, the end displacements of point (B), (u_B, v_B) are found by using the derived stiffness matrix for the three values of (γ). The results are listed in **Table (1)**. Through the comparison of the two solutions, it is found that the results obtained by using the derived stiffness matrix is close to the exact solution obtained by **Just(1982)**.

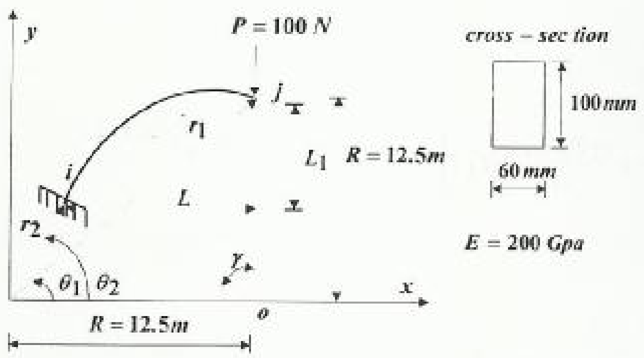

Fig.6: Curved Cantilever of The Numerical Example

In the above figure, the unknowns ($r_1, r_2, \theta_1, \theta_2$) can easily found from the geometry of the curved beam.

Table (1): Comparison of Results of The Numerical Example

End Deflections (mm)	γ	Number of Straight Elements Just (1982)			Exact Just (1982)	Present Analysis
		1	2	3		
$v_B \downarrow$	90°	94.28	138.13	152.11	157.08	156.908
$u_B \rightarrow$		94.28	97.67	99.37	100.00	100.448
$v_B \downarrow$	60°	50.00	58.08	60.55	61.42	60.975
$u_B \rightarrow$		28.87	25.88	25.22	25.00	24.58
$v_B \downarrow$	30°	8.63	8.93	9.03	9.06	9.174
$u_B \rightarrow$		2.31	1.92	1.83	1.79	1.805

- CONCLUSIONS

The derived stiffness matrix is found to be efficient for the analysis of curved beams as shown by the comparison of results obtained from the present analysis and the exact solution. It is found that the difference between the present and the exact analyses is not more than (1.7%). The derived stiffness matrix car be used for a wide range of curved elements starting from circular curved elements to straight elements. In addition, the global coordinates system is considered in the derivation of stiffness matrix, so, it can be used without any transformation. This will offer an economical time and fewer calculations solution than the derivations presented by different researchers.

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- NOTATIONS

The following symbols are used in this paper

A : cross-sectional area of the element

E : modulus of elasticity

I : the moment of inertia about the major axis

K_{ij} : coefficients of stiffness matrix $[K]$

L : the horizontal projection length of the curved element on $x-axis$ (span length)

L_1 : the vertical projection length of the curved element on $y-axis$

M : internal moment at a point with (r,θ) coordinates on the curved element

P : internal axial force at a point with (r,θ) coordinates on the curved element

Q : internal shear force at a point with (r,θ) coordinates on the curved element

u_i, u_j : horizontal displacements at the node (i) and (j) respectively

v_i, v_j : vertical displacements at the node (i) and (j) respectively

Appendix-A (Numerical Integration)

One of the most formulas which is used for numerical integration is(**Gaussian Quadrature**) formula . To estimate the value of the integration ($\int_{\theta_1}^{\theta_2} f(\theta)\,d\theta$) according to (**Gaussian Quadrature**) formula, the interval of the integration will be changed from $[\theta_1,\theta_2]$ to $[-1,1]$ by a suitable transformation of variable. Let the new variable (α), where $-1 \le \alpha \le 1$, be defined by

$$\alpha = \frac{2\theta-(a+b)}{b-a} \qquad \text{(A-1)}$$

Also define a new function $F(\alpha)$ so that

$$F(\alpha) = f(\theta) = f(\frac{(b-a)\alpha+(b+a)}{2}) \qquad \text{(A-2)}$$

Then, integration of $F(\alpha)$ between the integration limit $[-1,1]$ can be found as follows

$$\int_{-1}^{1} F(\alpha)\,d\alpha = \sum_{i=1}^{n} w_i F(\alpha_i) \qquad \text{(A-3)}$$

Where n is the number of Gaussian points. So, according to eq.(A-3) and by the subistitutions

$$\theta = \frac{(a+b)+(b-a)\alpha}{2}, and, \quad d\theta = \frac{(b-a)}{2}\,d\alpha$$

$$\int_{a}^{b} f(\theta)\,d\theta = \frac{(b-a)}{2}\left[\sum_{i=1}^{n} w_i f(\frac{(a+b)+(b-a)\alpha_i}{2})\right] \qquad \text{(A-4)}$$

Where (w_i) is the weight factor and (α_i) is the corresponding base point.

Example:

Find $A = \int_{\pi/4}^{\pi/2} (\cos\theta)^4\,d\theta$

a) Exact solution

$$A = \left[\frac{1}{4}(\cos\theta)^3 \sin\theta + \frac{3}{4}(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\right]_{\pi/4}^{\pi/2}$$

$= 0.04447$

b) Approximate solution

Using three Gussian points, the corresponding base point and weight factors are:

$\alpha_1 = 0.0$ $w_1 = 0.8888888$ $\theta_1 = 1.178$

$\alpha_2 = +0.77459$ $w_2 = 0.5555555$ $\theta_2 = 1.482$

$\alpha_3 = -0.77459$ $w_3 = 0.5555555$ $\theta_3 = 0.874$

Hence,

$$A = \int_{\pi/4}^{\pi/2} (\cos\theta)^4 \, d\theta = \frac{(b-a)}{2}\left[\sum_{i=1}^{3} w_i (\cos\theta_i)^4\right] = 0.3927[0.88888(0.021466) + 0.55555(0.16969)]$$

$A = 0.0445$

It can be seen that the two solutions are very close to each other.

Table (A-1): The values of the appropriate base points and the corresponding weight factors for $n = 1,2,3,.....,6$ points formula

Roots(α_i)	$\int_{-1}^{1} F(\alpha)d\alpha = \sum_{i=1}^{n} w_i F(\alpha_i)$	Weight factors (w_i)
+0.57735 02691 89626	Two- point Formula	1.0000000000000000000
-0.57735 02691 89626		1.0000000000000000000
0.0000000000000000	Three- point Formula	0.8888888888888888888
+0.77459 66692 41483		0.5555555555555555555
-0.77459 66692 41483		0.5555555555555555555
+0.33998 10435 84856	Four- point Formula	0.65214 51548 62546
-0. 33998 10435 84856		0.65214 51548 62546
+0.86113 63115 94053		0.34785 48451 37454
-0.86113 63115 94053		0.34785 48451 37454
0.0000000000000000	Five- point Formula	0.56888 88888 88889
+0.53846 93101 05683		0.47862 86704 99366
-0.53846 93101 05683		0.47862 86704 99366
+0.90617 98459 38664		0.23692 68850 56189
-0.90617 98459 38664		0.23692 68850 56189
+0.23861 91860 83197	Six- point Formula	0.46791 39345 72691
-0.23861 91860 83197		0.46791 39345 72691
+0.66120 93864 66256		0.36076 15730 48139
-0.66120 93864 66256		0.36076 15730 48139
+0.93246 95142 03152		0.17132 44923 79170
-0.93246 95142 03152		0.17132 44923 79170