OPTIMAM SHAPE OF TAPERED COLUMNS UNDER AXIAL COMPRESSIVE FORCE

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ABSTRACT:

The continuous and rapid development of shapes for architectural members and structural researches development in stability field encourage the designers and researchers to study the effect of member shape on the load capacity. The optimum strength of compression members with economical dimensions can be obtained to satisfy the architectural and technical requirements. A different nonlinear tapered member shapes have been studied under compression axial force with simply supported end conditions. Then graphical comparisons for different member shapes are presented to find the maximum axial force and minimum member volume at buckling state. This study is based on modified stability functions that have been based on Bessel functions. The results of this study provide structural and architectural designers the most proper member's shape, with more economical dimensions to carry the design load.

ألخلاصه:

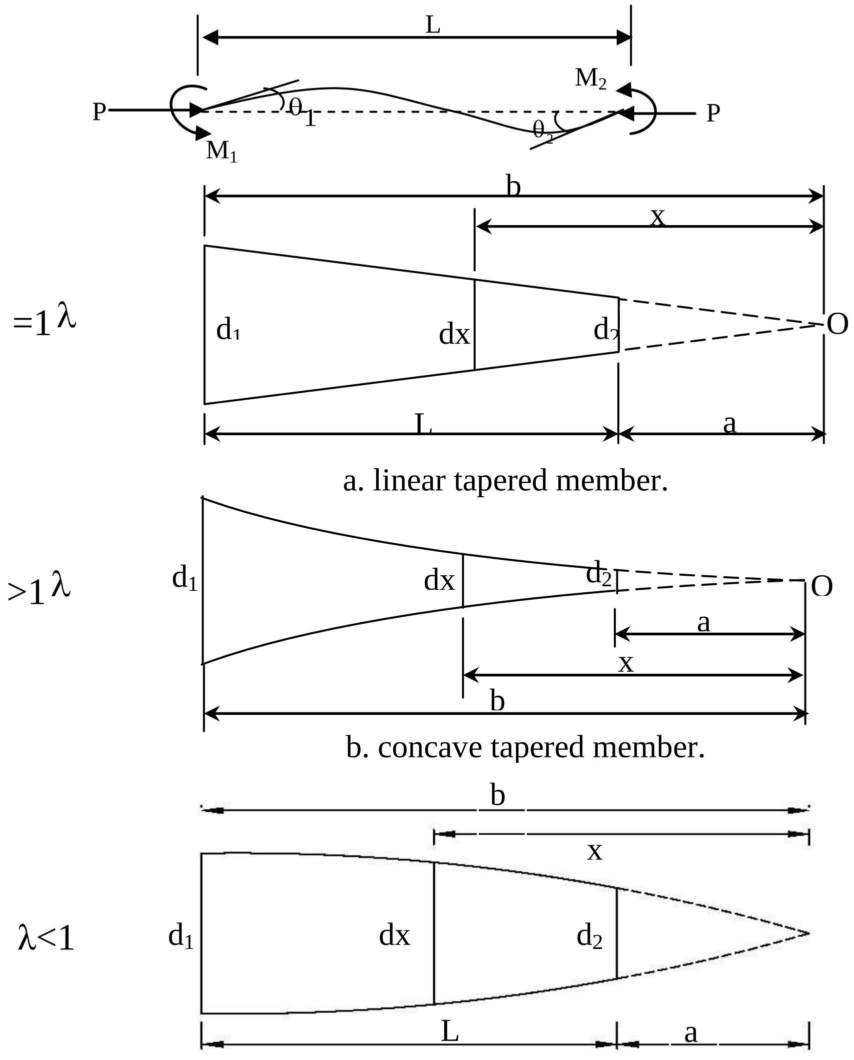
إن التطور المتزايد و المستمر في الشكل المعماري للعتبات الإنشائية و كذلك تطور البحوث في ميدان الاستقرارية حث الباحثين والمصممين على دراسة الشكل المعماري وتأثيره على قابلية تحمل العتبات للاحمال المحورية لايجاد التحمل الاقصى لهذه العتبات والذي يحقق الجانب الاقتصادي في نفس الوقت مع توفر كافة المتطلبات المعمارية والفنية. وتم في هذه الدراسة تقديم انواع من العتبات متساوية الطول والحجم ومختلفة من حيث تخصر الشكل متضمنة التخصر المقعر والمحدب والخطي وهي نماذج العتبات التي تم دراستها تحت تأثير الأحمال المحورية ومسندة إسنادا بسيطا. ثم تم عرض منحنيات للمقارنة بين العتبات المختلفة توضح أقصى قوة محورية يمكن إن تحمّل واقل حجم مطلوب لهذه العتبات قبل حصول فشل الانبعاج.ان هذه الدراسة تمت بالاعتماد على دوال الاستقرارية المحدثة والتي تم اشتقاقها في البحوث السابقة وهذه الدوال تستند على الدوال الرياضية التي تدعى دوال بسل ذات النوع الأول من الدرجة n. أن نتائج هذه الدراسة تمكّن أي مصمم إنشائي أو معماري على اختيار العتبات ذات الشكل المناسب والإبعاد الأكثر اقتصادية وتحقق في نفس الوقت قدرة هذه العتبات على تحمل الحمل التخميني المطلوب.

KEY WORDS:

Stability function, Non-linearity shape, Tapered member, Bessel function, Concave, Convex and Axial load.

INTRODUCTION:

The model of this study is a tapered member in both dimensions. The section depth and width along the member length producing the linearly or non-linearly tapered member is shown in **Fig. (1)**. This model is subjected to compression axial force. Increasing the section dimensions will enhance the strength of the individual structural members and enhance the whole structure load capacity.



a. linear tapered member.

b. concave tapered member.

c. convex tapered member.

Fig. 1: Tapered beam-column member.

MODIFIED STABILITY FUNCTIONS:

The modified stability functions of non sway and non-prismatic members in different non-linear configuration shape as derived in previous researches by (AL-Damluji and Yossif 2005), (Al-Sarraf and Yossif 2005) and (Yossif 2006) can be written in the form of modified slope deflection equations as given in eq. (1) and eq. (2) below:

$$M_1 = \frac{EI_2}{L}\left(S_1\theta_1 + SC\,\theta_2\right) \qquad (1)$$

$$M_2 = \frac{EI_2}{L}\left(SC\,\theta_1 + S_2\,\theta_2\right) \quad (2)$$

These two equations can be written in matrix form as given in eq. (3).

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \frac{EI_2}{L}\begin{bmatrix} S_1 & SC \\ SC & S_2 \end{bmatrix} . \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad (3)$$

Therefore the stiffness matrix $[K]$ for simply supported beam-column member can be written as given in eq. (4).

$$[K] = \frac{EI_2}{L}\begin{bmatrix} S_1 & SC \\ SC & S_2 \end{bmatrix} \quad (4)$$

Where I_2 is the moment of inertia at smaller depth of member, S_1, SC and S_2 are the modified stability functions and L is the member length.

The derivation of the modified stability functions based on the exact solution given in eq. 5 of second order differential equation of the deflected shape of beam column member subjected to compression axial force and bending moments.

$$EI(x)\frac{d^2y}{dx^2} + Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \quad (5)$$

where I(x) is the moment of inertia at distance x from origin O that can be written with respect to member dimensions as given in eq. (6) and with respect to member depth as given in eq. (7): –

$$I(x) = I_2\left(x/a\right)^{\overline{m}} \quad (6)$$

$$I(x) = d(x)^4/12 \quad (7)$$

Where $\overline{m}$ is the modified shape factor as obtaine from eq. (8). Otherwise the equation of shape factor can be obtained using eq. (9) (Al-Sarraf , 1979) and (Al-Damarchi, 1999) which depends on logarithm ratio of moment of inertia between two ends and tapering ratio: –

$$\overline{m} = \lambda m \quad (8)$$

$$m = \log\left(I_2/I_1\right)/\log u \quad (9)$$

In fact, the non-linear shape of non-prismatic members can be classified in three types according to the non-linearity factor λ:

1. $\lambda > 1$, for concave tapered member along its axis.
2. $\lambda = 1$, for linear tapered member along its axis.
3. $\lambda < 1$, for convex tapered member along its axis.

Where u and $\overline{u}$ are the tapering ratio and modified tapering ratio respectively that can be obtained from eq. (10), eq. (11) and eq. (12): –

$$\overline{u} = a/b \quad (10)$$

$$\bar{u} = u^{\frac{1}{\lambda}} \tag{11}$$

$$u^{\frac{1}{\lambda}} = a/b \tag{12}$$

Where b and a are the distance from origin O to end 1 and end 2 respectively as shown in Fig. (1). The depth d (x) may be expressed by eq. (13):

$$d(x) = d_2 (x/a)^{\lambda} \tag{13}$$

From eq. (13), the depth d_1 at end 1 when x = b can be obtained from eq. (14):

$$d_1 = d_2 (b/a)^{\lambda} \tag{14}$$

The moment of inertia of the strut may be expressed in the form shown in eq. (15):

$$I(x) = I_2 (x/a)^{\lambda m} \tag{15}$$

The basic differential equation of beam- column as given in eq. (5) can be written as in eq. (16) after substituting eq. (8) and eq. (15) into eq. (5).

$$EI_2 \left(\frac{x}{a}\right)^{\bar{m}} \frac{d^2y}{dx^2} + Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \tag{16}$$

The right hand side of eq. (16) can be reduced to zero by replacing "y" by "Z" when

$$Z = y - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b). \tag{17}$$

Thus, the differential equation becomes:

$$\frac{d^2Z}{dx^2} + \omega^2 x^{-\bar{m}} Z = 0 \tag{18}$$

Where

$$\omega^2 = Qa^{\bar{m}} / EI_2 . \tag{19}$$

Eq. (18) can be transformed into Bessel Equation of the form shown in eq. (20) (McLachla, 1961):

$$\frac{d^2Z}{dx^2} - \frac{(2\alpha - 1)}{x} \cdot \frac{dZ}{dx} + \left(\beta^2 \gamma^2 x^{2\gamma-2} + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right) Z = 0. \tag{20}$$

This equation has a general solution (McLachlan, 1961):

$$Z = x^{\alpha} \left[AJ_n (\beta x^{\gamma}) + BJ_{-n} (\beta x^{\gamma}) \right] \tag{21}$$

Depending on (n) is not an integer; here J_n is the Bessel functions of the first kind of order n; A and B are the constants of integration. Therefore, the solution can be written down in terms of Bessel functions by giving particular values to the constants α, β, γ and n, by comparing the two equations eq. (18) and eq. (20), the constants α, β, γ and n can be obtained: –

$$\alpha = 0.5\ ,\ \beta = \pm\frac{2\omega}{2-\overline{m}}\ ,\ \gamma = \frac{2-\overline{m}}{2}\ ,\ n = \pm\frac{1}{2-\overline{m}}$$

Hence, the general solution of the fundamental eq. (16) is:

$$y = \sqrt{x}\left[AJ_n\left(\beta x^{\gamma}\right) + BJ_{-n}\left(\beta x^{\gamma}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \qquad (22)$$

There are four unknowns A, B, M_1, and M_2, which have to be determined from the following boundary conditions:

at $x = a$, deflection $y = 0$ and rotation $dy/dx = \theta_2$,
and $x = b$, deflection $y = 0$ and rotation $dy/dx = \theta_1$.

The solution of the basic differential equation as given in eq. (22) can be represented by the modified stability functions for each type of nonlinear shape as given in **Table (1)** (Abdul Mahdi, 2002), (Al-Azawi 2005) and (Yossif, 2006).

Table (1): Modified stability functions for different non-linearity shape (Yossif, 2006).

	m	**λ**	$\overline{m}$	**S_1**	**SC**	**S_2**
Concave	4	0.2	0.8	$\frac{(\omega Lf_4 - Za^{0.4})}{[\omega PEI_2/LZQb^{0.4}]}$	$\frac{(\omega Lf_5 - Za^{-0.1}b^{0.5})}{[-\omega PEI_2/LZQa^{0.5}b^{-0.1}]}$	$\frac{(\omega Lf_3 - Zb^{0.4})}{[-\omega PEI_2/LZQa^{0.4}]}$
	4	0.4	1.6	$\frac{(\omega Lf_4 - Za^{0.8})}{[\omega PEI_2/LZQb^{0.8}]}$	$\frac{(\omega Lf_5 - Za^{0.3}b^{0.5})}{[-\omega PEI_2/LZQa^{0.5}b^{0.3}]}$	$\frac{(\omega Lf_3 - Zb^{0.8})}{[\omega PEI_2/LZQa^{0.8}]}$
	4	0.6	2.4	$\frac{(\omega Lf_4 + Za^{1.2})}{[-\omega PEI_2/LZQb^{1.2}]}$	$\frac{(\omega Lf_5 + Za^{0.5}b^{0.7})}{[\omega PEI_2/LZQa^{0.7}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{1.2})}{[-\omega PEI/LZQa^{1.2}]}$
	4	0.8	3.2	$\frac{(\omega Lf_4 + Za^{1.6})}{[-\omega PEI_2/LZQb^{1.6}]}$	$\frac{(\omega Lf_5 + Za^{0.5}b^{1.1})}{[\omega PEI_2/LZQa^{1.1}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{1.6})}{[-\omega PEI_2/LZQa^{1.6}]}$
Linear	4	1	4	$\frac{(\omega Lf_4 + Za^{2})}{[-\omega PEI_2/LZQb^{2}]}$	$\frac{(\omega Lf_5 + Za^{0.5}b^{1.5})}{[\omega PEI_2/LZQa^{1.5}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{2})}{[-\omega PEI_2/LZQa^{2}]}$
Convex	4	1.4	5.6	$\frac{(\omega Lf_4 + Za^{2.8})}{[\omega PEI_2/LZQb^{2.8}]}$	$\frac{(\omega Lf_5 - Za^{0.5}b^{2.3})}{[\omega PEI_2/LZQa^{2.3}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{2.8})}{[\omega PEI_2/LZQa^{2.8}]}$

	4	1.8	7.2	$\frac{(\omega Lf_4 + Za^{3.6})}{[\omega PEI_2/LZQb^{3.6}]}$	$\frac{(\omega Lf_5 - Za^{0.5}b^{3.1})}{[\omega PEI_2/LZQa^{3.1}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{3.6})}{[\omega PEI_2/LZQa^{3.6}]}$
	4	2.2	8.8	$\frac{(\omega Lf_4 + Za^{4.4})}{[-\omega PEI_2/LZQb^{4.4}]}$	$\frac{(\omega Lf_5 + Za^{0.5}b^{3.9})}{[\omega PEI_2/LZQa^{3.9}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{4.4})}{[-\omega PEI_2/LZQa^{4.4}]}$
	4	2.6	10.2	$\frac{(\omega Lf_4 + Za^{5.2})}{[-\omega PEI_2/LZQb^{5.2}]}$	$\frac{(\omega Lf_5 + Za^{0.5}b^{4.7})}{[\omega PEI_2/LZQa^{4.7}b^{0.5}]}$	$\frac{(\omega Lf_3 + Zb^{5.2})}{[-\omega PEI_2/LZQa^{5.2}]}$

Where any symbol in the above equations is defined in appendix A1, A2 andA3

ELASTIC CRITICAL LOAD:

A compression force Q if loaded axially on any beam-column member by a small value that is applied through the centroid of the cross section, the column remains straight and undergoes only axial compression. This straight form of equilibrium is stable, which means that the column returns to the straight position if it is disturbed. For instance, if a small lateral load is applied which causes the column to bend, the deflection will disappear and the column will return to the original position when the lateral load is removed. As the axial load is gradually increased, it reaches a condition of neutral equilibrium in which the column may have a bent shape. The ideal column may undergo small lateral deflections with no change in the axial force, and a small lateral load will produce a bent shape that does not disappear when the lateral load is removed. The stiffness matrix of beam-column member is gradually vanished at this load therefore such a load is called the elastic critical load.

The elastic critical load Q_c of any tapered member shape can be obtained using eq. (23) as given below:

$$Q_c = \rho . Q_E \, . \qquad (23)$$

Where: ρ_c is non-dimensional axial force parameter at the elastic critical load.

Q_E is the equivalent Euler load for tapered member which equal to $EI_2\pi^2/L^2$

E is modules of elasticity for steel

I_2 is the moment of inertia at smaller end depth

L is member length

The elastic critical load is obtained for linear tapered members and five types of tapering ratio which equal to 1.5, 2, 3, 4 and 5, then tabulated with respect to the properties ratio I_2/L^2 as given in **Table (2)** and presented graphically as shown in **Fig. (2)**.

The value of non-dimensional axial force parameter at the elastic critical load can be obtained using stability equations that given in **Table (1).**

Table (2): Elastic critical load for different depth at smaller depth's end

I_2/L	**Elastic Critical Load Q (kN)**				
	b/a=1.5	**b/a=2.0**	**b/a=3.0**	**b/a=4.0**	**b/a=5.0**
0.000833333	3.70	6.58	14.80	26.32	41.12
0.013333333	59.22	105.28	236.87	421.10	657.97
0.067500000	299.79	532.96	1199.16	2131.83	3330.99
0.213333333	947.48	1684.41	3789.93	6737.65	10527.58
0.520833333	2313.19	4112.34	9252.75	16449.34	25702.09
1.080000000	4796.63	8527.34	19186.51	34109.35	53295.86

2.000833333	8886.35	15797.95	35545.38	63191.79	98737.17
3.413333333	15159.71	26950.60	60638.85	107802.40	168441.25
5.467500000	24282.93	43169.65	97131.71	172678.60	269810.31
8.333333333	37011.02	65797.36	148044.07	263189.45	411233.52
12.20083333	54187.83	96333.92	216751.32	385335.67	602086.99
17.28000000	76746.04	136437.41	306984.18	545749.64	852733.82
23.80083333	105707.16	187923.85	422828.66	751695.39	1174524.05
32.01333333	142181.52	252767.15	568726.08	1011068.59	1579794.68
42.18750000	187368.27	333099.15	749473.08	1332396.59	2081869.68
54.61333333	242555.40	431209.60	970221.59	1724838.38	2695059.98
69.60083333	309119.71	549546.15	1236478.84	2198184.61	3434663.45
87.48000000	388526.85	690714.39	1554107.39	2762857.58	4316964.97
108.6008333	482331.27	857477.81	1929325.07	3429911.24	5359236.31
133.3333333	592176.26	1052757.80	2368705.06	4211031.21	6579736.27

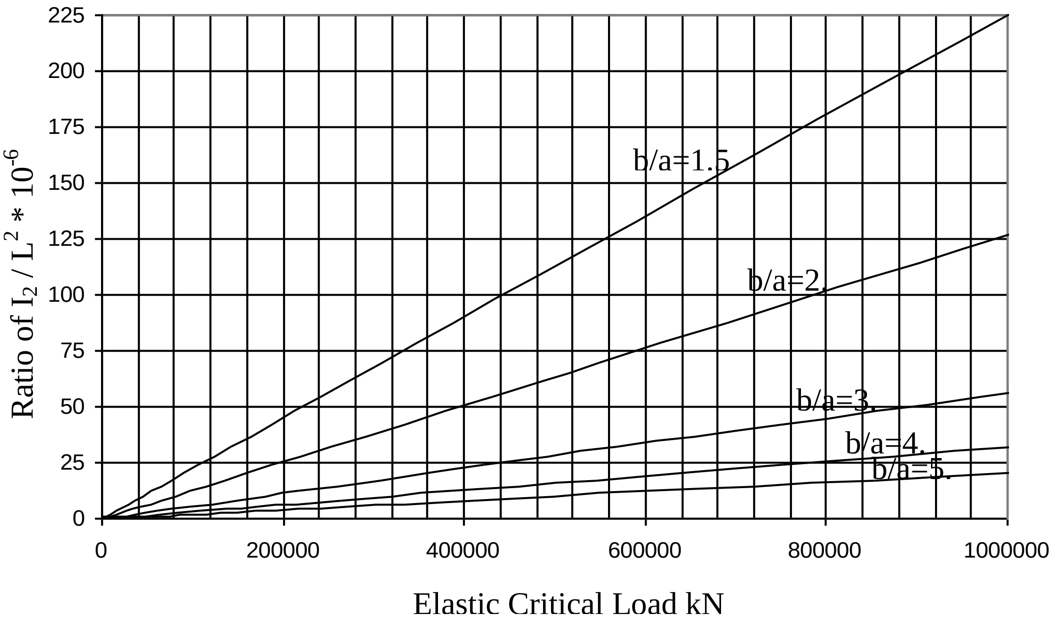


Figure (2): Elastic critical load for different tapering ratio of linear tapered shape

Then the modified stability function values substituted into the matrix given in eq. (4) with different compressive axial force value until the equation of stiffness matrix becomes equal to zero. At this case the axial load is the elastic critical load. The non-dimensional axial force parameter at elastic critical load are tabulated for nine different non-linearity shape and five tapering ratio as given in **Table (3)**.

Table (3): Non-dimensional Axial Force Parameter

λ	b/a=1.5	b/a=2.0	b/a=3.0	b/a=4.0	b/a=5.0
0.2	1.190	1.364	1.685	1.982	2.263
0.4	1.408	1.821	2.727	3.69	4.716
0.6	1.657	2.358	4.232	5.995	8.978
0.8	1.935	3.134	6.297	10.461	15.611
1.0	2.250	4.000	9.000	16.000	25.000

1.4	2.991	6.219	16.518	32.005	52.706
1.8	3.892	9.139	27.001	54.812	92.567
2.2	4.963	12.793	40.532	84.453	144.561
2.6	6.212	17.198	56.088	12.902	203.633

ECONOMICAL AND OPTIMUM SHAPE:

The economical member shape in this study represents the member that has a minimum material weight i.e. minimum member volume while the optimum member shape can be defined as the member which has minimum shape volume with maximum load capacity. The optimum shape is not necessarily being the economical member shape.

The optimum member shape depends on the optimum degree ratio represented by the ratio between member capacity and member volume. That mean the member shape convergence to the case of optimum shape when the optimum degree increased.

Nine different shapes of tapered beam-column members are presented in this study assigned according to the degree of non-linearity factor. General volume equations for different cross section area of tapered member derived as a function of non-linearity degree as given in **Table (4)** that can be used to compare between volumes of different member shapes having the same capacity of axial load.

Table (4): General volume equation

Section Type	Volume Function
Square section	$\frac{d_2^{\,2}\left(b^{2\lambda+1}-a^{2\lambda+1}\right)}{a^{2\lambda}\left(2\lambda+1\right)}$
Rectangular section	$\frac{\psi_2.d_2^{\,2}\left(b^{2\lambda+1}-a^{2\lambda+1}\right)}{a^{2\lambda}\left(2\lambda+1\right)}$
Circular section	$\frac{d_2^{\,2}.\pi.\left(b^{2\lambda+1}-a^{2\lambda+1}\right)}{4.a^{2\lambda}\left(2\lambda+1\right)}$

The ratio of the elastic critical loads of different non-linearity shape from the elastic critical loads of linear tapered member at tapering ratio u=2 are presented in **Table (5)** as the first numerical comparison between different tapered member shapes having the same other properties.

Table (5): Ratio of elastic critical load from linear tapered shape

λ	**b/a=1.5**	**b/a=2**	**b/a=3**	**b/a=4**	**b/a=5**
0.2	1.35930	1.35456	1.34557	1.33771	1.33122
0.4	1.34392	1.30339	1.24151	1.18318	1.13540
0.6	1.31595	1.20208	1.06945	0.87873	0.84358
0.8	1.27325	1.12498	0.86182	0.67749	0.54964
1	1.22161	1.00000	0.65237	0.44444	0.31868
1.4	1.09224	0.73125	0.31682	0.15197	0.08113
1.8	0.94111	0.48686	0.12854	0.04121	0.01585

2.2	0.78294	0.29878	0.04549	0.00950	0.00259
2.6	0.63044	0.17111	0.01426	0.00195	0.00037

The comparison presented in **Table (5)** explains that the convex shape has a more axial load capacity than other shapes in which it will reach about to 136% from the linear shape capacity; on the other hand the concave shape is less axial load capacity than other shapes in which it will be reach less than 0.04% from the linear shape capacity.

The elastic critical load of different non-linearity shape as a ratio from linear tapered shape is presented as the first comparison in graphical curve for different tapering ratio as shown in **Fig. (3)** for members having the same member length, volume and support conditions.

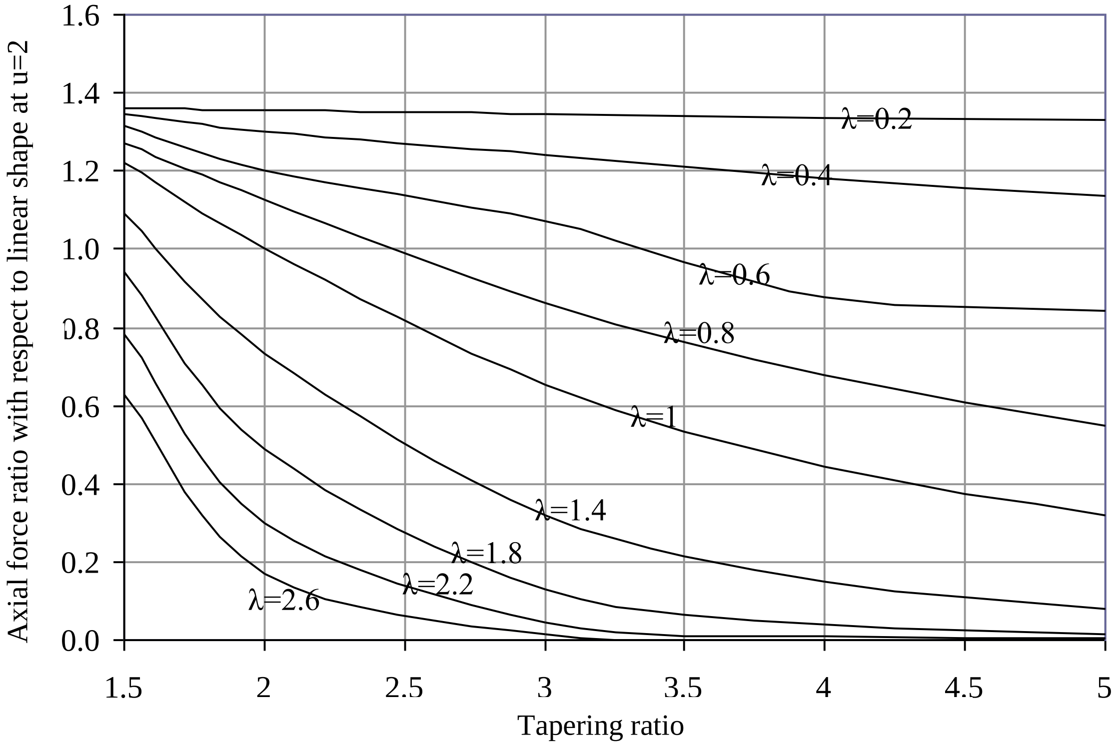


Figure (3): Comparison of models according to elastic critical loads

The ratio of members volume from the volume of linear tapered member at tapering ratio u=2 is presented in **Table (6)** as the second numerical comparison for different non-linearity shape and different tapering ratio.

Table (6): Volume ratio with respect to linear shape at u=2

Tapering ratio, u	**λ=0.2**	**λ=0.4**	**λ=0.6**	**λ=0.8**	**λ=1**	**λ=1.4**	**λ=1.8**	**λ=2.2**	**λ=2.6**
1.5	0.86	0.86	0.87	0.89	0.90	0.96	1.03	1.13	1.26
2	0.86	0.88	0.91	0.94	1.00	1.17	1.43	1.83	2.42
3	0.86	0.90	0.97	1.08	1.24	1.77	2.79	4.68	8.37
4	0.86	0.92	1.07	1.21	1.50	2.56	4.92	10.25	22.63
5	0.87	0.94	1.09	1.35	1.77	3.51	7.94	19.61	52.16

The comparison of **Table (6)** explains that the convex shape needs more volume than other shapes to carry the same axial load capacity which can be reached to 5216% as that of the linear shape. On the other hand the concave shape needs less volume than other shapes to carry the same axial load capacity which can be reached to 86% as that of the linear shape.

The volume for members having the same length and the same elastic critical load capacity with different tapering ratios is presented as the second comparison in another graphical curve as shown in **Fig. (4)**.

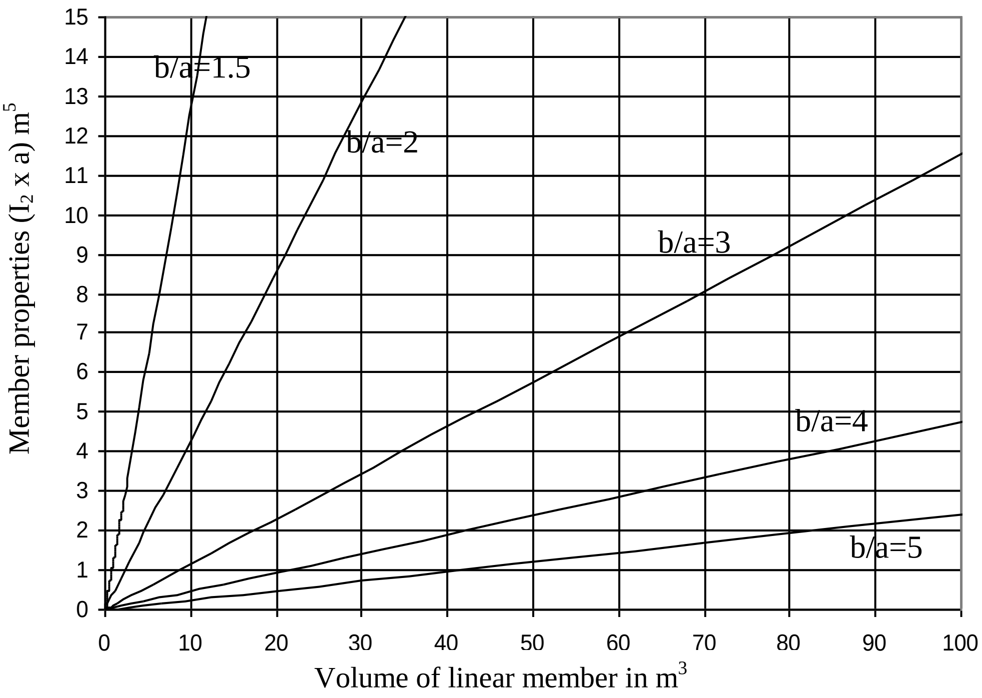


Figure (4): Comparison of models according to member volume

APPLICATIONS:-

The three simply supported steel beam-column members subjected to a compressive axial force with equal lengths of 4.0 m and solid rectangular cross sections tapered in different configuration shapes:

1-linear tapered member

2-concave tapered member with non-linearity equal to 2.6

3-convex tapered member with non-linearity equal to 0.2

The members have the same dimensions at the smaller depth's end which is equal to 5 cm for members depth and width, and the same tapering ratio of 3.

The comparison of the elastic critical force and volume for the three members are tabulated to verify the optimum member's shape:

First member:

The elastic critical load of this member can be obtained using eq. (23) and **Table (3)** as given below:

$$Q = \rho.\frac{EI_2\pi^2}{L^2} = 9.\frac{200000000\,x\,0.05^4/12\,x\,\pi^2}{4^2} = 578.3\text{kN}$$

The volume of this beam-column can be obtained using **Table (5)** as given below:

$$V=\frac{d_2{}^2.\left(b^{2\lambda+1}-a^{2\lambda+1}\right)}{a^{2\lambda}\left(2\lambda+1\right)}=\frac{0.05^2.\left(6^{2\times1+1}-2^{2\times1+1}\right)}{2^{2\times1}\left(2\times1+1\right)}=0.043\ m^3$$

Second member:

The elastic critical load of this member can be obtained using eq. (23) and **Table (3)** as given below:

$$Q=\rho.\frac{EI_2\pi^2}{L^2}=56.088.\frac{200000000\,x\,0.05^4/12x\,\pi^2}{4^2}=3604\,kN$$

The volume of this beam-column can be obtained using **Table (5)** as given below:

$$V=\frac{d_2{}^2.\left(b^{2\lambda+1}-a^{2\lambda+1}\right)}{a^{2\lambda}\left(2\lambda+1\right)}=\frac{0.05^2.\left(6^{2\times2.6+1}-2^{2\times2.6+1}\right)}{2^{2\times2.6}\left(2\times2.6+1\right)}=0.732\ m^3$$

Third member:

The elastic critical load of this member can be obtained using eq. (23) and **Table (3)** as given below:

$$Q=\rho.\frac{EI_2\pi^2}{L^2}=1.685.\frac{200000000\,x\,0.05^4/12\,x\,\pi^2}{4^2}=108.3\,kN$$

The volume of this beam-column can be obtained using **Table (5)** as given below:

$$V=\frac{d_2{}^2.\left(b^{2\lambda+1}-a^{2\lambda+1}\right)}{a^{2\lambda}\left(2\lambda+1\right)}=\frac{0.05^2.\left(6^{2\times0.2+1}-2^{2\times0.2+1}\right)}{2^{2\times0.2}\left(2\times0.2+1\right)}=0.013\,m^3$$

The above results are tabulated according to the non-linearity degree as given in **Table (8)**:

Table (8): Application Results

	Concave λ=2.6	Linear λ=1.0	Convex λ=0.2
Elastic critical load Q, kN	3604	578.3	108.3
Elastic critical load Q, %	623%	100%	18.7%
Volume V, m^3	0.732	0.043	0.013
Volume V, %	1702%	100%	30%
$\frac{Q\%}{V\%}$	36.6%	100%	62.3%

CONCLUSIONS:

This study is based on mathematical solution of nine different models of tapered members having square or circular solid cross-sectional area to find mathematical functions which named modified stability functions which are used to obtain the elastic critical load for each member shape.

The results of this study are presented by two comparison tables and graphical curves. The maximum shape strength is obtained at the non-linearity factor of 2.6 “concave shape” with tapering ratio of 1.5 which is equal to 136% of the linear shape strength. The minimum shape strength is also found at the non-linearity 0.2 “convex shape” with tapering ratio of 5 which is less than 0.04% of the linear shape strength. Then the more economical shape is found at the non-linearity degree of 0.2 and tapering ratio of 1.5 in which it requires about 86% of the linear shape volume to carry the same axial load, on the other hand the least economical shape is found at the non-linearity degree of 2.6 and tapering ratio of 5 which is required about 5216% of the linear shape volume to carry the same load.

Another comparison can be resulted from the application using load capacity-volume ratio in which the first optimum degree is obtained in the non-linearity degree equal to 1"linear shape", and the less optimum degree when the non-linearity value convergence to 1.

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SYMBOLS:

d_1 and d_2 : Depth at end 1 and end 2 respectively
$f_1, f_2, \ldots f_6$: Parameters function
K : Stiffness of the strut
$m, \overline{m}$: Shape factor and modified Shape factor respectively
n : The order of Bessel function
y : Deflection
u : Tapering ratio
$\bar{u}$: Modified tapering ratio
E : Young's modulus
I_1 , I_2 : Moment of inertia at end 1and end 2 respectively
L : Member length
M_1, M_2 : Bending moments at end 1 and 2 of the member
Q : Axial load

Q_c : Elastic critical load
Q_E : Eular load = EI_2L^2/π^2
S_1, S_2 : Modified stability function at end 1and end 2 respectively
SC : Carrying factor of the modified stability function
λ : Non-linearity factor
θ_1, θ_2 : Rotation at end 1 and 2 of member due to bending moment and axial force
ρ : Non-dimensional axial force parameter
ρ_c : Elastic critical load parameter
ψ_2 : Ratio of member cross sectional dimensions (width to depth) at smaller depth

APPENDIX A

Table A-1: Symbols of Stability Equations for members having λ=0.2, 0.4 and 0.6

Symbol	$\bar{m}=0.8 \quad \lambda=0.2$	$\bar{m}=1.6 \quad \lambda=0.4$	$\bar{m}=2.4 \quad \lambda=0.6$
Z	$J_{-0.166}(\alpha)J_{0.166}(\beta)-J_{0.166}(\alpha)J_{-0.166}(\beta)$	$J_{-2.5}(\alpha)J_{0.2.5}(\beta)-J_{2.5}(\alpha)J_{-2.5}(\beta)$	$J_{2.5}(\alpha)J_{-2.5}(\beta)-J_{-2.5}(\alpha)J_{2.5}(\beta)$
ω	$\left(a^{0.8}Q/EI_2\right)^{0.5}$	$\left(a^{1.6}Q/EI_2\right)^{0.5}$	$\left(a^{2.4}Q/EI_2\right)^{0.5}$
α	$1.667\omega a^{0.6}$	$5\omega a^{0.2}$	$5\omega a^{-0.2}$
β	$1.667\omega b^{0.6}$	$5\omega b^{0.2}$	$5\omega b^{-0.2}$
A	$-\frac{M_1J_{-0.167}(\alpha)\sqrt{a}+M_2J_{-0.167}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$-\frac{M_1J_{-2.5}(\alpha)\sqrt{a}+M_2J_{-2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{-2.5}(\alpha)\sqrt{a}+M_2J_{-2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
B	$\frac{M_1J_{0.167}(\alpha)\sqrt{a}+M_2J_{0.167}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{2.5}(\alpha)\sqrt{a}+M_2J_{2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$-\frac{M_1J_{2.5}(\alpha)\sqrt{a}+M_2J_{2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
f_1	$J_{-1.166}(\alpha)J_{1.166}(\beta)-J_{1.166}(\alpha)J_{-1.166}(\beta)$	$J_{-3.5}(\alpha)J_{3.5}(\beta)-J_{3.5}(\alpha)J_{-3.5}(\beta)$	$J_{-3.5}(\alpha)J_{3.5}(\beta)-J_{3.5}(\alpha)J_{-3.5}(\beta)$
f_2	$J_{-0.166}(\alpha)J_{0.166}(\beta)-J_{0.166}(\alpha)J_{-0.166}(\beta)$	$J_{-2.5}(\alpha)J_{2.5}(\beta)-J_{2.5}(\alpha)J_{-2.5}(\beta)$	$J_{-2.5}(\alpha)J_{2.5}(\beta)-J_{2.5}(\alpha)J_{-2.5}(\beta)$
f_3	$J_{-0.166}(\alpha)J_{1.166}(\beta)+J_{0.166}(\alpha)J_{-1.166}(\beta)$	$J_{-2.5}(\alpha)J_{3.5}(\beta)+J_{2.5}(\alpha)J_{-3.5}(\beta)$	$J_{-2.5}(\alpha)J_{3.5}(\beta)+J_{2.5}(\alpha)J_{-3.5}(\beta)$
f_4	$J_{-0.166}(\beta)J_{1.166}(\alpha)+J_{0.166}(\beta)J_{-1.166}(\alpha)$	$J_{-2.5}(\beta)J_{3.5}(\alpha)+J_{2.5}(\beta)J_{-3.5}(\alpha)$	$J_{-2.5}(\beta)J_{3.5}(\alpha)+J_{2.5}(\beta)J_{-3.5}(\alpha)$
f_5	$J_{0.166}(\beta)J_{-1.166}(\beta)+J_{-0.166}(\beta)J_{1.166}(\beta)$	$J_{2.5}(\beta)J_{-3.5}(\beta)+J_{-2.5}(\beta)J_{3.5}(\beta)$	$J_{2.5}(\beta)J_{-3.5}(\beta)+J_{-2.5}(\beta)J_{3.5}(\beta)$
f_6	$J_{-0.166}(\alpha)J_{1.166}(\alpha)+J_{0.166}(\alpha)J_{-1.166}(\alpha)$	$J_{-2.5}(\alpha)J_{3.5}(\alpha)+J_{2.5}(\alpha)J_{-3.5}(\alpha)$	$J_{-2.5}(\alpha)J_{3.5}(\alpha)+J_{2.5}(\alpha)J_{-3.5}(\alpha)$
P1	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$
P2	$\left(f_4b^{0.5}-f_6a^{0.5}\right)$	$\left(f_4b^{0.5}-f_6a^{0.5}\right)$	$\left(f_4b^{0.5}-f_6a^{0.5}\right)$
P	$Z\left[-a^{-0.1}P_1-b^{-0.1}P_2\right]-\omega Lf_1f_2$	$Z\left[b^{0.3}P_2-a^{0.3}P_1\right]-\omega Lf_1f_2$	$Z\left[a^{0.7}P_1-b^{0.7}P_2\right]-\omega Lf_1f_2$

Table A-2: Symbols of Stability Equations for members having λ=0.8, 1.0 and 1.4

Symbol	$\overline{m}=3.2 \quad \lambda=0.8$	$\overline{m}=4 \quad \lambda=1$	$\overline{m}=5.6 \quad \lambda=1.4$
Z	$J_{0.833}(\alpha)J_{-0.833}(\beta)-J_{-0.833}(\alpha)J_{0.833}(\beta)$	$J_{0.5}(\alpha)J_{-0.5}(\beta)-J_{-0.5}(\alpha)J_{0.5}(\beta)$	$J_{0.78}(\alpha)J_{-0.278}(\beta)-J_{-0.278}(\alpha)J_{0.278}(\beta)$
ω	$\left(a^{3.2}Q/EI_2\right)^{0.5}$	$\left(a^{4}Q/EI_2\right)^{0.5}$	$\left(a^{5.6}Q/EI_2\right)^{0.5}$
α	$0.56\omega a^{-1.8}$	ω/a	$0.556\omega a^{-1.8}$
β	$1.667\omega b^{-0.6}$	ω/b	$0.56\omega b^{-1.8}$
A	$\frac{M_1J_{-0.833}(\alpha)\sqrt{a}+M_2J_{-0.833}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{-0.5}(\alpha)\sqrt{a}+M_2J_{-0.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{-0.278}(\alpha)\sqrt{a}+M_2J_{-0.278}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
B	$-\frac{M_1J_{0.833}(\alpha)\sqrt{a}+M_2J_{0.833}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$-\frac{M_1J_{0.5}(\alpha)\sqrt{a}+M_2J_{0.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$-\frac{M_1J_{0.279}(\alpha)\sqrt{a}+M_2J_{0.278}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
f_1	$J_{-1.833}(\alpha)J_{1.833}(\beta)-J_{1.833}(\alpha)J_{-1.833}(\beta)$	$J_{-1.5}(\alpha)J_{1.5}(\beta)-J_{1.5}(\alpha)J_{-1.5}(\beta)$	$J_{-1.278}(\alpha)J_{1.278}(\beta)-J_{1.278}(\alpha)J_{-1.278}(\beta)$
f_2	$J_{-0.833}(\alpha)J_{0.833}(\beta)-J_{0.833}(\alpha)J_{-0.833}(\beta)$	$J_{-0.5}(\alpha)J_{0.5}(\beta)-J_{0.5}(\alpha)J_{-0.5}(\beta)$	$J_{-0.278}(\alpha)J_{0.278}(\beta)-J_{0.278}(\alpha)J_{-0.278}(\beta)$
f_3	$J_{-0.833}(\alpha)J_{1.833}(\beta)+J_{0.833}(\alpha)J_{-1.833}(\beta)$	$J_{-0.5}(\alpha)J_{1.5}(\beta)+J_{0.5}(\alpha)J_{-1.5}(\beta)$	$J_{-0.278}(\alpha)J_{1.278}(\beta)+J_{0.278}(\alpha)J_{-1.278}(\beta)$
f_4	$J_{-0.833}(\beta)J_{1.833}(\alpha)+J_{1.833}(\beta)J_{-1.833}(\alpha)$	$J_{-0.5}(\beta)J_{1.5}(\alpha)+J_{0.5}(\beta)J_{-1.5}(\alpha)$	$J_{-0.278}(\beta)J_{1.278}(\alpha)+J_{0.278}(\beta)J_{-1.278}(\alpha)$
f_5	$J_{-0.833}(\beta)J_{1.833}(\beta)+J_{0.833}(\beta)J_{-1.833}(\beta)$	$J_{0.5}(\beta)J_{-1.5}(\beta)+J_{-0.5}(\beta)J_{1.5}(\beta)$	$J_{0.278}(\beta)J_{-1.278}(\beta)+J_{-0.278}(\beta)J_{1.278}(\beta)$
f_6	$J_{-0.833}(\alpha)J_{1.833}(\alpha)+J_{1.833}(\alpha)J_{-1.833}(\alpha)$	$J_{-0.5}(\alpha)J_{1.5}(\alpha)+J_{0.5}(\alpha)J_{-1.5}(\alpha)$	$J_{-0.278}(\alpha)J_{1.278}(\alpha)+J_{0.278}(\alpha)J_{-1.278}(\alpha)$
P1	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$
P2	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$
P	$Z\left[a^{0.5}P_1-b^{0.5}P_2\right]-\omega Lf_1f_2$	$Z\left[a^{1.5}P_1-b^{1.5}P_2\right]-\omega Lf_1f_2$	$Z\left[a^{2.3}P_1-b^{2.3}P_2\right]-\omega Lf_1f_2$

Table A-3: Symbols of Stability Equations for members having λ=1.8, 2.2 and 2.6

Symbol	$\overline{m}=7.2 \quad \lambda=1.8$	$\overline{m}=8.8 \quad \lambda=2.2$	$\overline{m}=10.4 \quad \lambda=2.6$
Z	$J_{-0.192}(\alpha)J_{0.192}(\beta)-J_{0.192}(\alpha)J_{-0.192}(\beta)$	$J_{0.147}(\alpha)J_{-0.147}(\beta)-J_{-0.147}(\alpha)J_{0.147}(\beta)$	$J_{0.119}(\alpha)J_{-0.119}(\beta)-J_{-0.119}(\alpha)J_{0.119}(\beta)$
ω	$\left(a^{7.2}Q/EI_2\right)^{0.5}$	$\left(a^{8.8}Q/EI_2\right)^{0.5}$	$\left(a^{10.4}Q/EI_2\right)^{0.5}$
α	$0.385\omega a^{-2.6}$	$0.294\omega a^{-3.4}$	$0.238\omega a^{-4.2}$
β	$0.385\omega b^{-2.6}$	$0.294\omega b^{-3.4}$	$0.238\omega b^{-4.2}$
A	$\frac{M_1J_{-0.192}(\alpha)\sqrt{a}+M_2J_{-0.192}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{-0.147}(\alpha)\sqrt{a}+M_2J_{-0.147}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{-0.119}(\alpha)\sqrt{a}+M_2J_{-0.119}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$

B	$-\frac{M_1J_{0.192}(\alpha)\sqrt{a}+M_2J_{0.192}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ}$	$-\frac{M_1J_{0.147}(\alpha)\sqrt{a}+M_2J_{0.147}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ}$	$-\frac{M_1J_{0.119}(\alpha)\sqrt{a}+M_2J_{0.119}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ}$
f_1	$J_{-1.192}(\alpha)J_{1.192}(\beta)-J_{1.192}(\alpha)J_{-1.192}(\beta)$	$J_{-1.147}(\alpha)J_{1.147}(\beta)-J_{1.147}(\alpha)J_{-1.147}(\beta)$	$J_{-1.119}(\alpha)J_{1.119}(\beta)-J_{1.119}(\alpha)J_{-1.119}(\beta)$
f_2	$J_{-0.192}(\alpha)J_{0.192}(\beta)-J_{0.192}(\alpha)J_{-0.192}(\beta)$	$J_{-0.147}(\alpha)J_{0.147}(\beta)-J_{0.147}(\alpha)J_{-0.147}(\beta)$	$J_{-0.119}(\alpha)J_{0.119}(\beta)-J_{0.119}(\alpha)J_{-0.119}(\beta)$
f_3	$J_{-0.192}(\alpha)J_{1.192}(\beta)+J_{0.192}(\alpha)J_{-1.192}(\beta)$	$J_{-0.147}(\alpha)J_{1.147}(\beta)+J_{0.147}(\alpha)J_{-1.147}(\beta)$	$J_{-0.119}(\alpha)J_{1.119}(\beta)+J_{0.119}(\alpha)J_{-1.119}(\beta)$
f_4	$J_{-0.192}(\beta)J_{1.192}(\alpha)+J_{0.192}(\beta)J_{-1.192}(\alpha)$	$J_{-0.147}(\beta)J_{1.147}(\alpha)+J_{0.147}(\beta)J_{-1.147}(\alpha)$	$J_{-0.119}(\beta)J_{1.119}(\alpha)+J_{0.119}(\beta)J_{-1.119}(\alpha)$
f_5	$J_{0.192}(\beta)J_{-1.192}(\beta)+J_{-0.192}(\beta)J_{1.192}(\beta)$	$J_{0.147}(\beta)J_{-1.147}(\beta)+J_{-0.147}(\beta)J_{1.147}(\beta)$	$J_{0.119}(\beta)J_{-1.119}(\beta)+J_{-0.119}(\beta)J_{1.119}(\beta)$
f_6	$J_{-0.192}(\alpha)J_{1.192}(\alpha)+J_{0.192}(\alpha)J_{-1.192}(\alpha)$	$J_{-0.47}(\alpha)J_{1.147}(\alpha)+J_{0.147}(\alpha)J_{-1.147}(\alpha)$	$J_{-0.119}(\alpha)J_{1.119}(\alpha)+J_{0.119}(\alpha)J_{-1.119}(\alpha)$
P1	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$
P2	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$
P	$Z[a^{3.1}P_1-b^{3.1}P_2]-\omega Lf_1f_2$	$Z[a^{3.9}P_1-b^{3.9}P_2]-\omega Lf_1f_2$	$Z[a^{4.7}P_1-b^{4.7}P_2]-\omega Lf_1f_2$