# STABILITY ANALYSIS OF SLOPES USING A DOUBLE SLIDING MODEL

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**ABSTRACT**

Experience with Bishop's method, using the shear strength parameters c and $\phi$ obtained from triaxial compression tests seems to indicate that the values of the safety factor obtained in this way are relatively large. This is sometimes compensated by using shear strength parameters obtained from direct shear tests, simple shear tests, or classical cell tests, which usually lead to smaller values of c and $\phi$. A solution for the difficulty mentioned above may be the notion that the simple Coulomb formula does not take into account that failure may occur on a plane perpendicular to the global slip surface, with an additional rotation to result in global slip parallel to the slip surface, i.e. double sliding failure model.

The essential step in developing the basic formula for many slope stability methods (Fellenius, Bishop, Janbu) is to derive an expression for the normal stress on the slip surface. In the Bishop method this is done by combining the equation of vertical equilibrium in which it is assumed that there is no net contribution of the shear forces on the sides of the slice, with the Coulomb equation.

The alternative mechanism proposed is that failure occurs not because the shear stress on the slip surface reaches the maximum value described by the Coulomb criterion, but that the shear stress on a plane perpendicular to the slip surface (and thus also the shear stress on a plane parallel to the slip surface) reaches the maximum value on this perpendicular plane, and thus, using a safety factor F.

In this paper, a comparison is made of the factor of safety calculated by the double sliding model with those calculated by Fellenius and Bishop's methods.

**الخلاصة**

لقد بينت الخبرة في تطبيق طريقة بيشوب في تحليل المنحدرات أن استعمال معاملات مقاومة القص c و $\phi$ المستحصلة من فحوص الانضغاط ثلاثي المحاور يبدو أنها تعطي قيما لمعاملات الأمان عالية نسبيا. و هذه المسألة يتم التغلب عليها أحيانا باستعمال معاملات مقاومة مستحصلة من فحوص القص المباشر و فحوص القص البسيط أو فحوص الخلية التقليدية التي تعطي عادة قيما أعلى للمعاملات c و $\phi$ . ان حل هذه المشكلة يمكن أن يتم من خلال إدراك أن معادلة كولومب لا تأخذ بنظر الاعتبار حقيقة أن الفشل يمكن أن يحدث على مستو عمودي على سطح الانزلاق الكلي مع دوران إضافي ينتج عنه انزلاق كلي مواز لسطح الانزلاق، و هذا يعني وجود نموذج لفشل الانزلاق المزدوج. ان الخطوة المهمة في تطوير المعادلة الأساسية لكثير من طرق تحليل المنحدرات (فيلينيوس، بيشوب و جامبو) تكمن في اشتقاق معادلة لإيجاد الإجهاد

العمودي على سطح الانزلاق. و في طريقة بيشوب تجرى هذه الخطوة بتركيب معادلة التوازن الشاقولي التي من خلالها يفترض أنه لا توجد مشاركة لقوى القص على جانبي الشريحة و التي تفترضها معادلة كولومب.
إن الآلية البديلة المقترحة هي أن الفشل يحدث ليس بسبب أن إجهاد القص على سطح الانزلاق يصل الى القيمة القصوى الموصوفة بمعادلة كولومب، و لكن بسبب أن إجهاد القص على مستو عمودي على سطح الانزلاق (و كذلك إجهاد القص على مستو مواز إلى سطح الانزلاق) يصل القيمة القصوى على هذا المستوي العمودي، و من خلاله يحسب معامل الأمان F .
في هذا البحث أجريت مقارنة بين معامل الأمان المحسوب بنموذج الانزلاق المزدوج و تلك المحسوبة بطريقة فيلينيوس و طريقة بيشوب.

**KEY WORDS**
Stability, Slopes, Bishop, Double, Sliding

## INTRODUCTION

The usual procedure in the analysis of stability of slopes is to calculate the safety factor of various assumed circular slip surfaces, and then to regard the slip surface having the smallest safety factor as critical. If the safety factor is smaller than 1 the slope is considered to be unstable. In normal conditions, the design of such a slope is rejected. In the design of dikes and dams it is usually required that the smallest safety factor is higher than 1,say 1.2 or 1.3. Such small safety factors have become accepted in the Netherlands because the shear strength parameters are often determined from cell tests, in which the ultimate failure state is not reached, but the shear strength parameters are determined as corresponding to a safe state of stress,with a certain small deformation rate, (Verruijt, 2002).

An alternative, which agrees better with international practice, is to use triaxial tests to determine the actual shear strength parameters, in combination with a standard stability analysis, such as Bishop's method. This will probably lead to much higher safety factors. In this paper Bishop's method (Bishop, 1955) is presented, with an extension to a double sliding failure criterion, which may reduce the safety factors.

## BISHOP'S METHOD

### Basic Principles

Bishop's simplified method is based upon a consideration of moment equilibrium of the soil mass above an assumed circular slip surface, see **Fig. (1)**. The soil mass is subdivided into a number of vertical slices, of width b and height h. The average unit weight in a slice is denoted by $\gamma$. The maximum shear stress acting at the lower boundary of a slice is related to the local cohesion c and the normal effective stress $\sigma_n'$ by Coulomb's relation:

$$\tau_f = c + \sigma_n' \tan\phi \qquad (1)$$

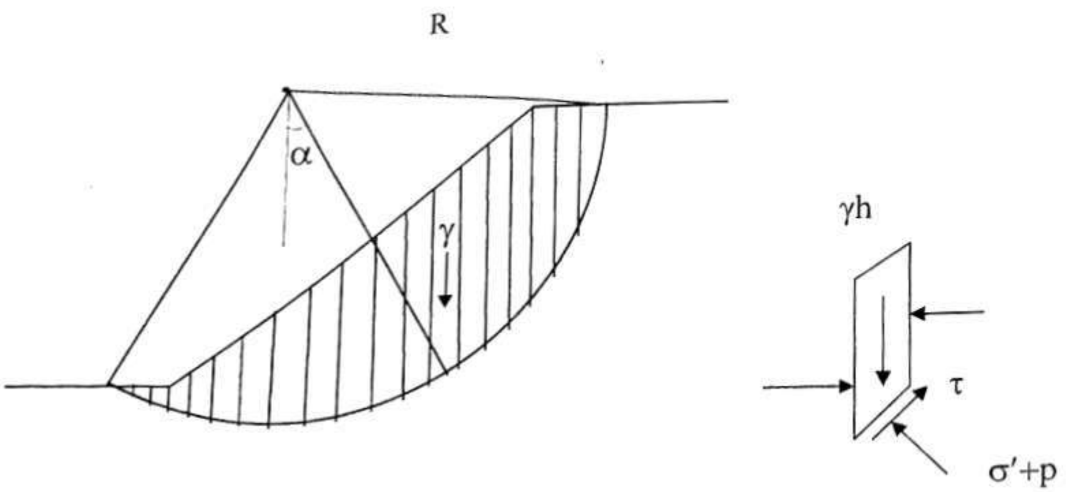
where $\phi$ is the angle of internal friction, c is the effective cohesion, and $\sigma_n'$ is the effective stress normal to the plane of failure.

It is assumed that the actual shear stress acting upon the lower boundary of a slice is $\tau_f / F$, where F is a certain constant, the stability factor ,or safety factor. Hence:

$$\tau = \frac{1}{F}(c + \sigma_n^{'} \tan\phi) \quad (2)$$

Equilibrium of moments with respect to the center of the slip circle can be expressed as follows:

$$\sum \gamma h b R \sin\alpha = \sum \frac{\tau \; bR}{\cos\alpha} \quad (3)$$




(a) slip surface (b) forces on a slice.

Fig. (1) Slip circle method.

If all slices have the same width, it follows from Equations (2) and (3) that:

$$F = \frac{\sum \left[(c + \sigma_n^{'} \tan\phi)/\cos\alpha\right]}{\sum \gamma h \sin\alpha} \quad (4)$$

This formula is the basis of several methods, such as those developed by Fellenius and Bishop (1955). Because Bishop's method has been validated against solutions for various particular cases and has been used extensively with satisfactory results, it is widely used in engineering practice.

In Bishop's method, it is assumed that the forces transmitted between adjacent slices are strictly horizontal. It then follows from the vertical equilibrium of a slice (see **Fig. 1**) that:

$$\gamma h = \sigma_n^{'} + p + \tau \tan\alpha \quad (5)$$

By using the expression (2) for the shear stress τ one now obtains:

$$\sigma_n^{'}(1 + \frac{\tan\alpha \tan\phi}{F}) = \gamma h - p - \frac{c}{F}\tan\alpha \quad (6)$$

It follows that:

$$c + \sigma_n^{'} \tan\phi = \frac{c/(\gamma h - p)\tan\phi}{1 + \tan\alpha \tan\phi / F} \quad (7)$$

Substitution of this expression into equation (4) for the stability factor F now gives finally:

$$F = \frac{\sum \frac{c + (\gamma h - p)\tan\phi}{\cos\alpha(1 + \tan\alpha\tan\phi / F)}}{\sum \gamma h \sin\alpha} \tag{8}$$

This is the basic formula of Bishop's method. Because the stability factor F also appears in the right hand side of the equation, its value must be determined iteratively, starting with an initial estimate. Experience has shown that the method usually converges very fast, and that the initial estimate can be taken as F =1 .0.

It should be noted that in the formula (8) the factor $\gamma h$ denotes the total weight of a slice of soil. In an inhomogeneous soil, this may be the sum of the weight of a number of sections consisting of different types of soil, from the top of the slice to its bottom. The shear strength parameters c and $\phi$ apply to the slip surface, that is the bottom of the slice.

**The modification by Koppejan**

The maximum shear stress acting at the bottom of a slice is given by, (Verruijt, 1994):

$$\tau_f = \frac{c + (\gamma h - p)\tan\phi}{1 + \tan\alpha\tan\phi / F} \tag{9}$$

If F =1 this shear stress becomes infinitely large for $\alpha = \phi - 1/2\pi$, because then tan $\alpha$ tan$\phi$ = -1. Such a value for the angle $\alpha$ may occur near the lower end of the slip circle, if the circle is deep, and the friction angle is not very large. For larger negative values of $\alpha$ the shear stress is negative, which would mean that the shear stress is not acting against the direction of slip. This may lead to unrealistic values for the stability factor, and therefore it has been suggested by Koppejan of Delft Geotechnics that the value of $\alpha$ to be used in the expression for the shear stress be cut off at $-1/4\pi + 1/2\pi$, which is one half of the critical value. This is called the modified Bishop method. In most cases, the cut off value is not reached, but it is a refinement that avoids unrealistic values for deep slip circles. This modification has been implemented in the program to be discussed below.

**Extension to a double sliding model**

Experience with Bishop's method, using the shear strength parameters c and $\phi$ obtained from triaxial compression tests seems to indicate that the values of the safety factor obtained in this way are relatively large. This is sometimes compensated by using shear strength parameters obtained from direct shear tests, simple shear tests, or classical cell tests, which usually lead to smaller values of c and $\phi$. This is an unsatisfactory situation, as the triaxial test in general is considered as superior to other tests, because the stress state at failure is completely known, and the test results usually are more accurate and less dependent on details of the test procedure such as the handling of the sample. Another difficulty may be that in the Netherlands it has become standard practice to use small critical (minimal) safety factors (say 1.2 or 1.3), perhaps because conservative values of the shear strength parameters were used. Applying less conservative values for the shear strength parameters of the soils, on the basis of triaxial tests, while maintaining the same minimum safety factors, would lead to less safety in engineering practice, even though engineering experience does not indicate that the safety levels of the dikes are too high.

A solution for the difficulty mentioned above may be the notion that the simple Coulomb formula (1) does not take into account that failure may occur on a plane perpendicular to the global slip surface, with an additional rotation to result in global slip parallel to the slip surface. This double sliding failure model was introduced by De Josselin de Jong (1971) (as cited by Verruijt, 2002).

The additional mechanism is illustrated in **Fig. (2)**. It can be considered as a generalization of the mechanism occurring in a row of books on a bookshelf, which may collapse if the horizontal support is not large enough.

The essential step in developing the basic formula for many slope stability methods (Fellenius, Bishop, Janbu) is to derive an expression for the normal stress on the slip surface. In the Bishop method this is done by combining the equation of vertical equilibrium (5), in which it is assumed that there is no net contribution of the shear forces on the sides of the slice, with the Coulomb equation (1).

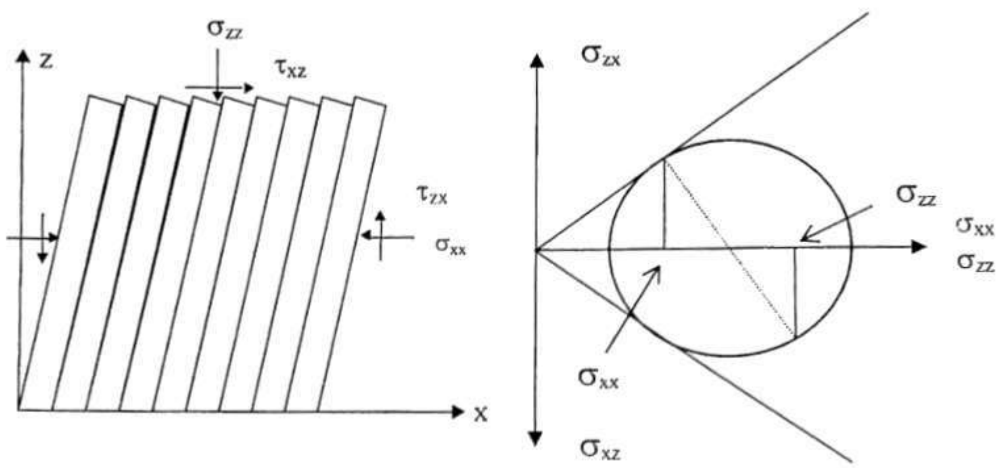



Fig. (2) Mechanism of a collapsing row of books.

The alternative mechanism proposed is that failure occurs not because the shear stress on the slip surface reaches the maximum value described by the Coulomb criterion, but that the shear stress on a plane perpendicular to the slip surface (and thus also the shear stress on a plane parallel to the slip surface) reaches the maximum value on this perpendicular plane, and thus, using a safety factor F:

$$\tau = \frac{1}{F}(c + \sigma_t^{'} \tan\phi) \tag{10}$$

where t is a tangential direction, perpendicular to the normal direction n .The expression for the safety factor now will be:

$$F = \frac{\sum\left[(c + \sigma_t^{'} \tan\phi)/\cos\alpha\right]}{\sum \gamma h \sin\alpha} \tag{11}$$

The difference with Equation (4) is that the normal stress in this expression is $\sigma'_t$, rather than $\sigma_n^{'}$. Equilibrium of the slice is formulated in the usual way by Equation (5). It is now assumed that the horizontal effective stress $\sigma_n^{'}$ is related to the vertical effective stress $\sigma_v^{'}$ by a horizontal stress coefficient $K_o$,

$$\sigma_h^{'} = K_o\sigma_v^{'} = K_o(\gamma h - p) \tag{12}$$

On the basis of the invariance of the isotropic part of the stress tensor for rotations of the coordinate system, a relation between the stresses $\sigma'_t$ and $\sigma'_n$ is:

$$\sigma_t^{'} + \sigma_n^{'} = \sigma_h^{'} + \sigma_v^{'} = (1 + K_o)\sigma_v^{'} = (1 + K_o)(\gamma h - p) \tag{13}$$

It now follows that:

$$\sigma_t^{'} = (1 + K_o)(\gamma h - p) - \sigma_n^{'} \tag{14}$$

or, with Equation (5):

$$\sigma_t' = K_o(\gamma h - p) - \tau \tan\alpha \tag{15}$$

With Equation (10) this gives:

$$\sigma_t'(1 - \tan\alpha \tan\phi / F) = K_o(\gamma h - p) + c\tan\alpha / F \tag{16}$$

Using this expression it follows that:

$$c + \sigma_t' \tan\phi = \frac{c + K_o(\gamma h - p)\tan\phi}{1 - \tan\alpha \tan\phi / F} \tag{17}$$

Substitution into Equation (11) gives:

$$F = \frac{\sum \dfrac{c + K_o(\gamma h - p)\tan\phi}{\cos\alpha(1 - \tan\alpha\tan\phi / F)}}{\sum \gamma h \sin\alpha} \tag{18}$$

This is the alternative value of the stability factor, assuming local failure along planes perpendicular to the global failure surface (the slip circle).

Again it is necessary to cut off the value of the shear stress at the slip surface, as proposed by Koppejan for the classical Bishop method, because the value of the factor (1-tan $\alpha$ tan$\phi$) will be zero if $\alpha = 1/2\pi - \phi$. For this reason, it is suggested that in this factor the value of $\alpha$ is never taken larger than $\alpha_{max} = 1/4\pi - 1/2\phi$.

As along the failure surface both of the two possibilities of local failure may occur, the final formula becomes:

$$F = \frac{\sum P}{\sum Q} \tag{19}$$

where:

$$P = \min\left[\frac{c + (\gamma h - p)\tan\phi}{\cos\alpha(1 + \tan\alpha\tan\phi / F)}\ ,\ \frac{c + K_o(\gamma h - p)\tan\phi}{\cos\alpha(1 - \tan\alpha\tan\phi / F)}\right] \tag{20}$$

and

$$Q = \gamma h \sin\alpha \tag{21}$$

In the factor (1 + tan $\alpha$ tan $\phi$/F) in Equation (20) the value of $\alpha$ should not be taken smaller than $\alpha_{min} = -1/4\pi + 1/2\phi$, and in the factor (1 - tan $\alpha$ tan $\phi$/F), the value of $\alpha$ should not be taken larger than $\alpha_{max} = 1/4\pi - 1/2\phi$.

## COMPUTER PROGRAM

The method described in this paper has been implemented in the computer program STB, which is a program for the analysis of stability of a slope using Bishop's simplified method, with some modifications introduced at GeoDelft at the Delft University.

The program STB contains two refinements of Bishop's method. The first refinement is that care is taken that the direction of the shear stress along the slip surface is always opposing the sliding mechanism. This is done by cutting off the value of $\alpha$ at a maximum value of ($\phi/2$ - $\pi/4$). The second refinement is that the shearing resistance is reduced if the coefficient of horizontal stress at rest (Ko) is so small that the slip would occur along a plane perpendicular to the slip surface, combined with a local rotation, in agreement with a double sliding model. This refinement is effective only if the coefficient of horizontal stress (Ko) is smaller than 1. If (Ko < 1), the program may reduce the shear strength on the slip surface by considering the possibility of local sliding in a thin zone along planes perpindicular to the slip surface, combined with a rotation.

## APPLICATION OF THE MODEL

The double sliding model is now applied to a problem of typical slope shown in **Fig. (3)**. This slope has been analysed and the factor of safety is calculated using three methods, namely: The Swedish circle (Fellenius) method, Bishop's method and the double sliding method. Different values of the coefficient of lateral stress at rest (Ko) are used. **Fig.(4)** shows the variation of the factors of safety calculated by the three methods with (Ko) for different values of the angle of friction ($\phi$).
It can be noticed that the factor of safety calculated by the double sliding model is always less than those calculated by Bishop's method for values of Ko < 1. These factors are smaller than those calculated by Fellenius method at some values of Ko depending on the value of $\phi$.

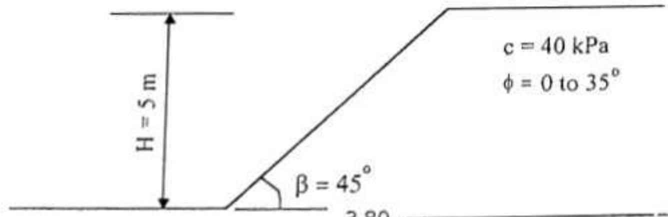
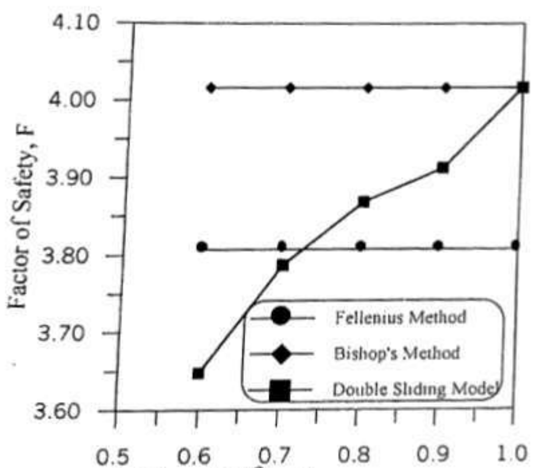
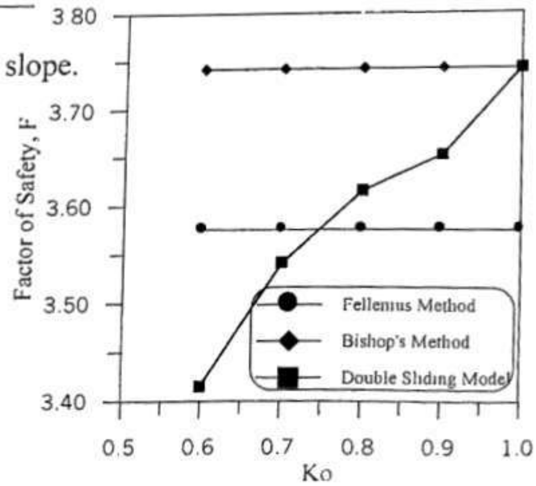


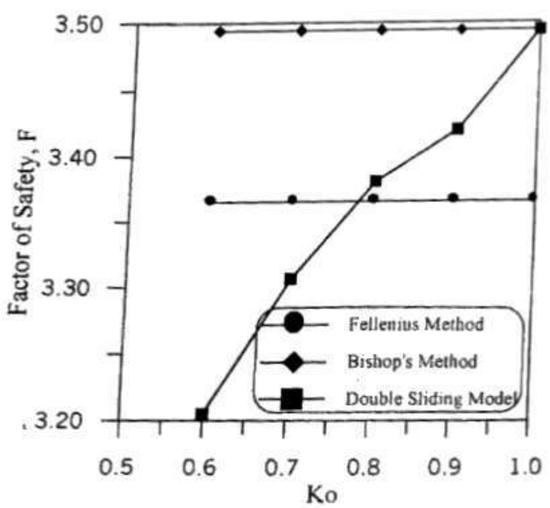

Fig. (3) – Typical slope.



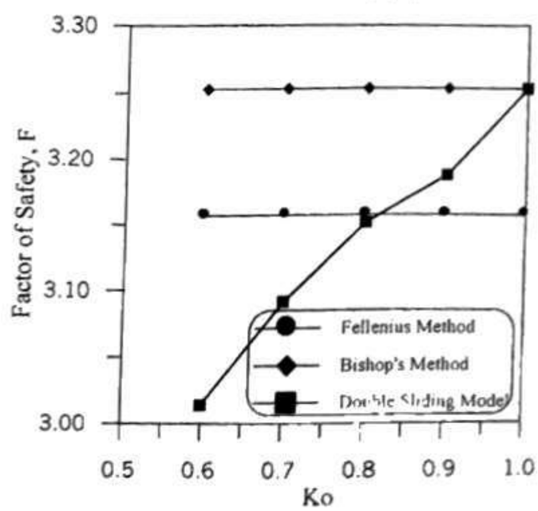

(a) $\phi = 35^{\circ}$ Ko




(b) $\phi = 30^{\circ}$

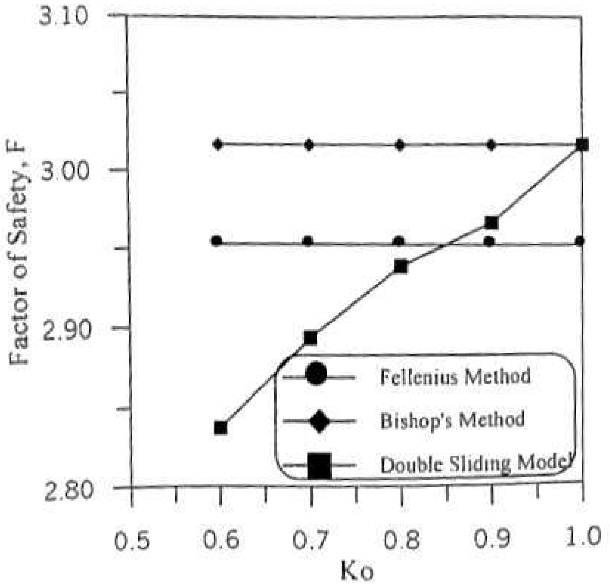



(c) $\phi = 25^{\circ}$

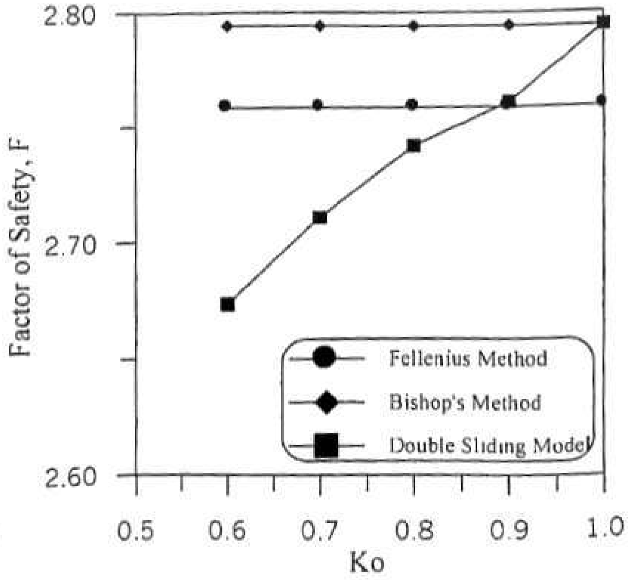



(d) $\phi = 20^{\circ}$

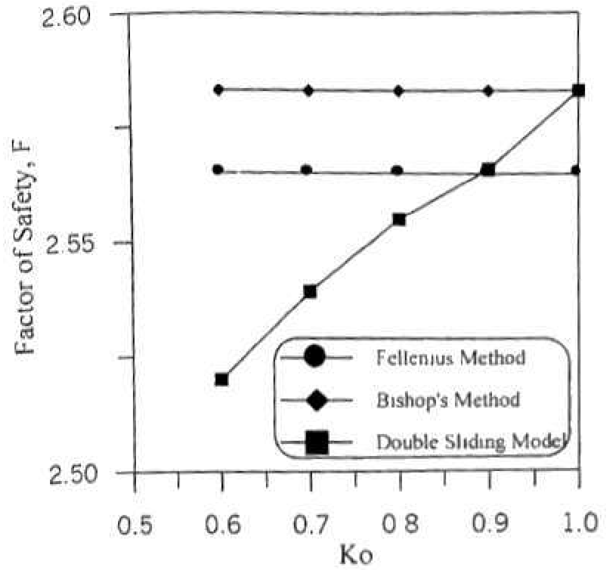
Fig. (4) A comparison between the factor of safety calculated by the double sliding model with those calculated by Fellenius and Bishop's methods.



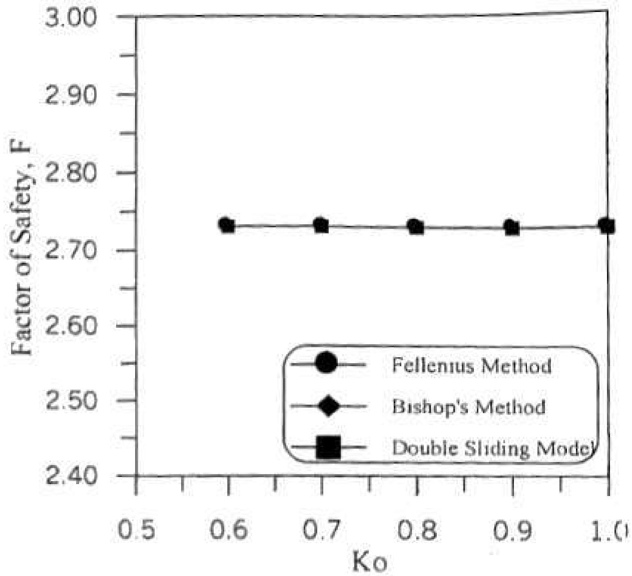

(e) $\phi = 15^o$




(f) $\phi = 10^o$




(g) $\phi = 5^o$




(h) $\phi = 0^o$

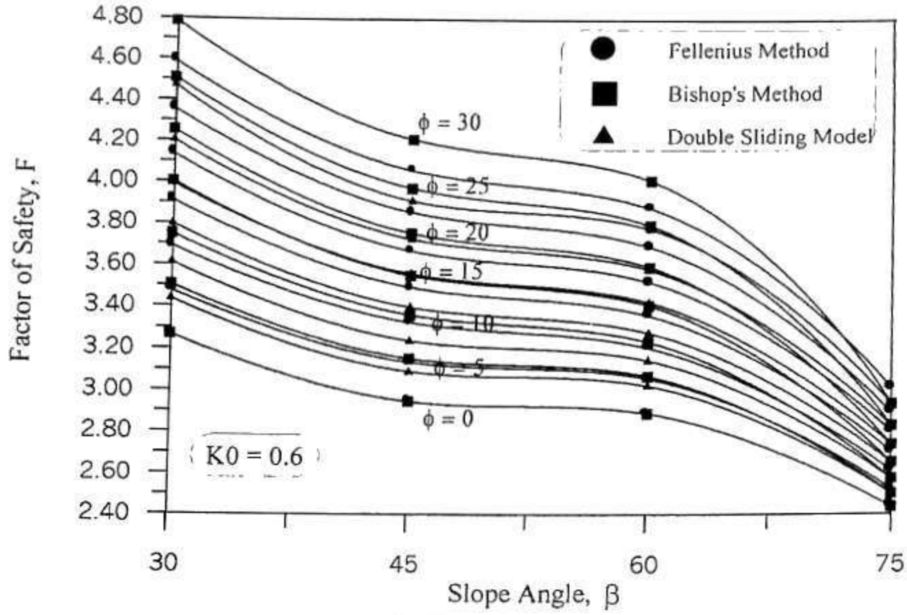
Fig. (4) – (Continued).

## EFFECT OF THE SLOPE ANGLE

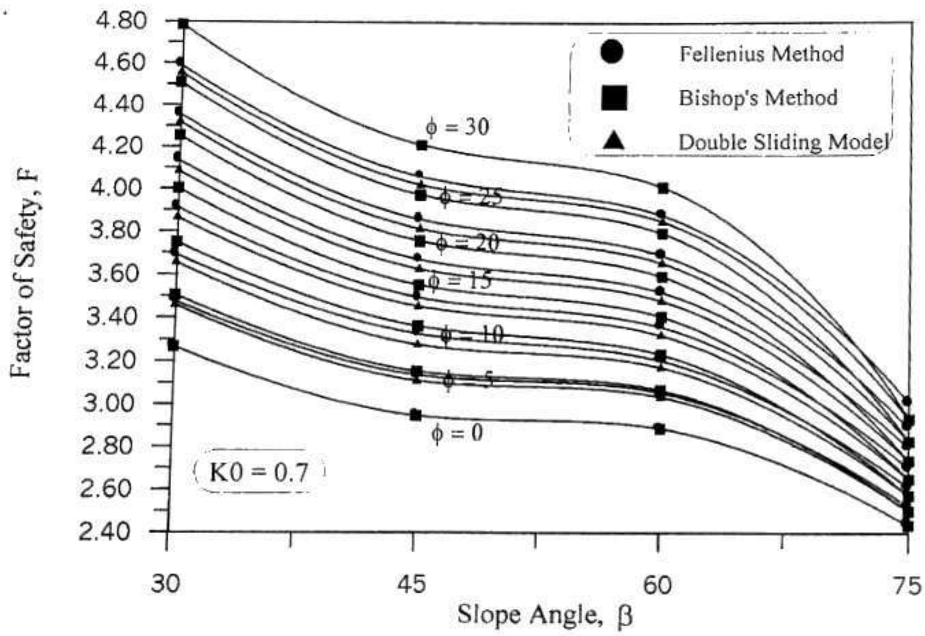
In order to verify the effect of the previous modifications on Bishop's method on the factor of safety of different slopes, the same typical slope of **Fig.(3)** is analysed using different slopes angles. The results are drawn in **Fig.(5)**.

It can be noticed in these figures that the values of the factor of safety by the three methods converge as the angle of friction decreases.

It can also be noticed that the differences between the factors of safety calculated by Bishop's method and the double sliding model decrease as the slope's angle increases, and the safety factors have the same values by the two methods when the slope angle ($\beta \geq 75^o$).

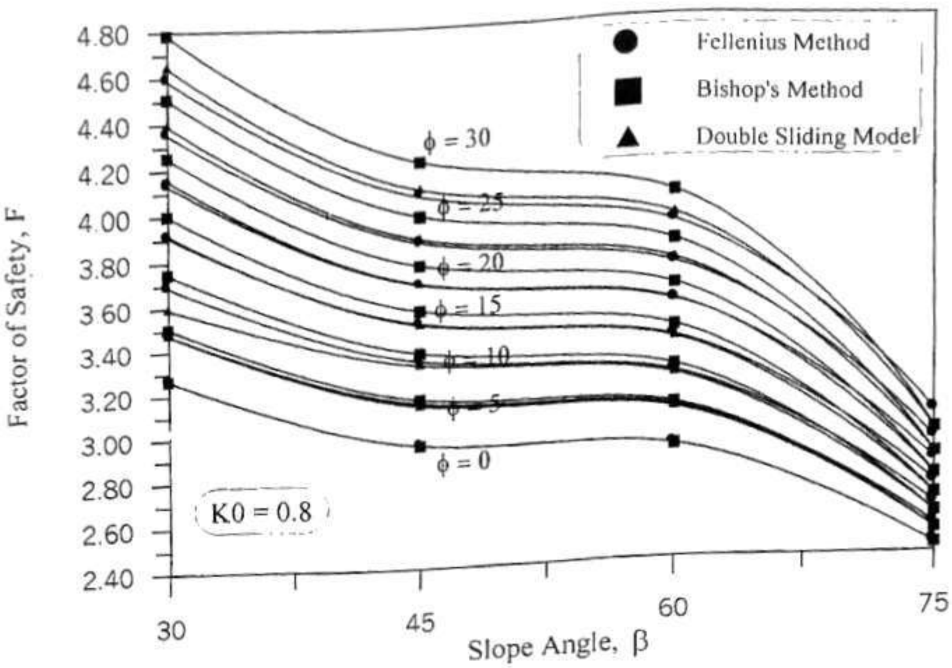



(a) Ko = 0.6

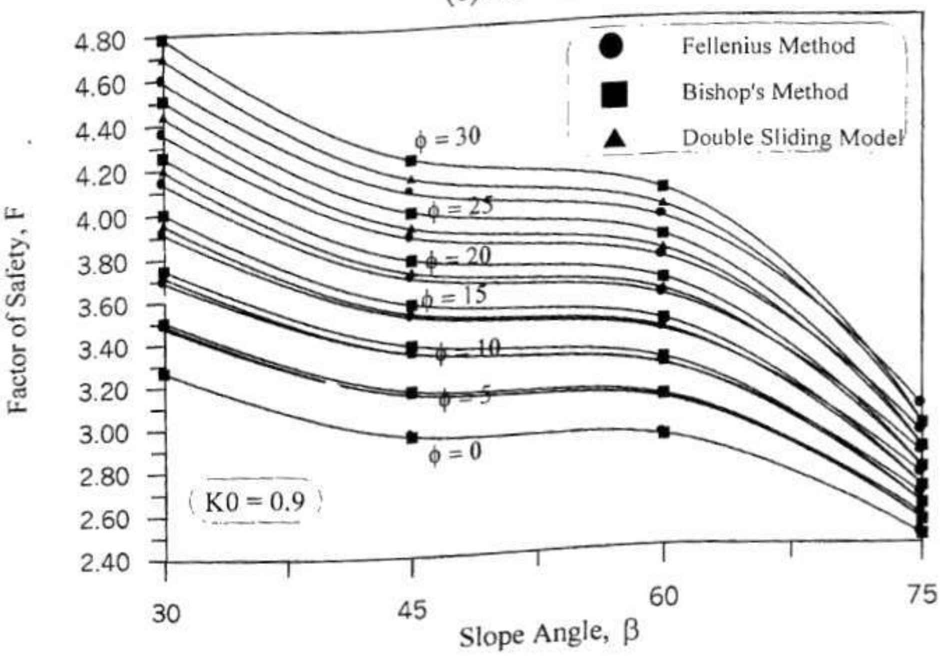



(b) Ko = 0.7

Fig. (5) The effect of slope angle on the factor of safety for different values of the coefficient of lateral stress at rest.

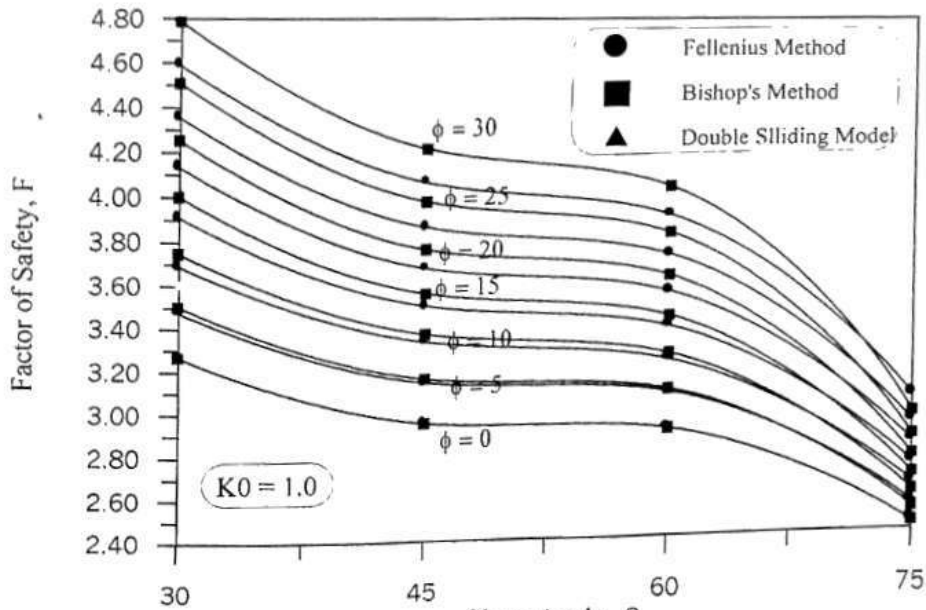



(c) Ko = 0.8




(d) Ko = 0.9

Fig. (5) – (Continued).




(e) Ko = 1.0

Fig. (5) – (Continued).

## CONCLUSIONS

In this paper Bishop's method of slices for slope stability is developed by applying the double sliding model. The mechanism of this model assumes that failure occurs not because the shear stress on the slip surface reaches the maximum value described by the Coulomb criterion, but that the shear stress on a plane perpendicular to the slip surface (and thus also the shear stress on a plane parallel to the slip surface) reaches the maximum value on this perpendicular plane, and thus, using a safety factor F.

From the analysis carried out in this paper, the following conclusions can be obtained:

1- The factor of safety calculated by the double sliding model is always less than those calculated by Bishop's method for values of Ko < 1. These factors are smaller than those calculated by Fellenius method at some values of Ko depending on the value of $\phi$. The values of the factor of safety by the three methods converge as the angle of friction decreases.

2- 2. The differences between the factors of safety calculated by Bishop's method and the double sliding model decrease as the slope's angle increases, and the safety factors have the same values by the two methods when the slope angle ($\beta \geq 75^{o}$).

## REFERENCES

Bishop, A. W., (1955), (The Use of the Slip Circle in the Stability Analysis of Slopes), Geotechnique, Vol. 5, p.p. 7-17.

Verruijt, A., (1994), Numerical Geomechanics, Delft University of Technology.

Verruijt, A., (2002), Slope Stability, Report from Internet, adapted from Computational

(Geomechanics), by A. Verruijt, 1995.