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STATIC ANALYSIS OF THIN-WALLED CURVED BEAM ELEMENT INCLUDING WARPING EFFECTS

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ABSTRA

A new mathematical model for three-dimensional thin-walled curved beam element of closed section with seven degree-of-freedom per node is derived using the finite element procedure. The seventh degree-of-freedom is to account for warping restraint effects in thin-walled closed sections. These effects may become significant and should be fully considered in such sections for which warping deformations are relatively large. This model considers the coupled action due to the curved geometry of the element using its exact static behavior in the derivation of the displacement field. Also, the model considers the non-uniform torsional behavior of closed thin-walled sections in cases where additional axial direct stresses and complementary shear stresses are formed. The developed warping function of this model considers the interaction between the normal warping stresses and the accompanying warping shear stresses as well as the coupled action between the torsion and bending.

In addition to the ordinary axial and flexural deformations, the strain energy, which is used to obtain the stiffness matrix of the developed curved beam element fully, considers the additional axial, primary and secondary shear deformations due to warping restraint. The validity of this element is investigated by comparing the developed program analysis results with some available analytical solutions.

الخلاصة

تم اشتقاق نموذج رياضي جديد لعنصر ذو مقطع نحيف الجدران من النوع المغلق ثلاثي الأبعاد ذو عقدتين وسبعة درجات حرية لكل عقدة باستخدام طريقة العناصر المحددة. تم استحداث درجة الحرية السابعة لإدخال تأثير التشوه الألتوائي للمقاطع المغلقة نحيفة الجدران. هذا التأثير يكون واضحا ويجب أن يؤخذ بنظر الاعتبار في مثل هذه المقاطع التي يكون تشوه الالتواء فيها كبير. استنادا الى التصرف الستاتيكي للعنصر عند اشتقاق مجال الإزاحة، يأخذ النموذج المقترح بنظر الاعتبار التأثير المزدوج الناتج عن شكل العنصر المنحني. إضافة الى ذلك يأخذ هذا النموذج بنظر الاعتبار التصرف الالتوائي غير المنتظم للمقاطع المغلقة عند تقييد ذلك التأثير مسببا" اجهادات محورية وقصية اضافية. دالة التشوه الالتوائي المشتقة لذلك العنصر تأخذ بنظر الاعتبار الترابط بين الاجهادات المحورية الناتجة عن ذلك التشوه واجهادات القص المصاحبة لها اضافة الى التأثير المزدوج بين عزم اللي وعزم الانحناء.

ان طاقة الانفعال المستخدمة في اشتقاق مصفوفة الجساءة للعنصر تتضمن، اضافة الى تأثير التشوه المحوري والانحناء الاعتيادي، تأثير التشوه المحوري والقص الأولي والثانوي الناتج عن التشوه الالتوائي. تم تطوير برنامج حاسوبي ولغرض التحقق من ملائمة دقة التصرف الساكن لهذا العنصر تم تقديم عدد من المسائل ومقارنة نتائج البرنامج المعد مع الحلول التحليلية المتوفرة.

KEY WORDS
Closed thin-walled, Curved beam element, Warping.

INTRODUCTION

Thin-walled sections are often used in bridges, highways and some other important structures. One of the most distinctive features of thin-walled sections is their response to torsional loading. If warping is restrained or non-uniform torsion is applied, out of plane warping will occur and additional normal shear stress will arise, therefore considering warping in the analysis of thin-walled structures is very important.

Some previous works consider the curvature of curved beam elements of closed sections when warping restraint effects are included (Yoo 1987& 1979). Castigliano's theorem is used to obtain the stiffness matrix by inverting the flexibility matrix. In the present work the finite element procedure is used to derive the natural shape functions of the new model which considers the coupled action due to the curved geometry of the element using the exact static behavior as well as the non-uniform torsional behavior of closed thin-walled sections when warping is taken as an explicit degree-of-freedom. Such a model will be useful in obtaining the stiffness as well as the consistent mass matrices, which can be used in the dynamic analysis of curved structures.

NON-UNIFORM TORSION IN BOX GIRDERS

When warping displacements vary along the length causing non-uniform distribution of torsion along it, because of restraints of warping displacement at a cross section or a varying applied torque, an additional normal stress and an accompanying shear stress develop. In thin-walled section there is an interaction between this additional normal stress and the warping shear deformation given by the second term of the following warping displacement equation:

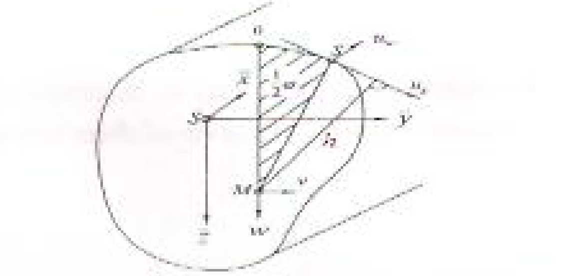
$$u_w = u_{wo} + \int_0^s \frac{T}{G\delta} ds - \tau' \int_0^s hds \tag{1}$$

u_{wo} is the displacement at $s = 0$.

Therefore the distribution of warping displacement u_w and warping normal stress σ_w at the cross-section is indeed affined to $\bar{\omega}$, but the relationship is not defined by τ' and τ'' , respectively as in the case of open sections, but instead by the first and second derivatives of a dimensionless warping function $f = f(x)$ (Dabrowski 1968).

$$u_w = -f'\bar{\omega} \tag{2}$$

$$\sigma_w = Eu'_w = -Ef''\bar{\omega} \tag{3}$$



Figure(3-9): The components of displacement due to torsional shear deformation

Fig. (1) The components displacement due to torsional shear deformatio

Thus the bimoment (BM), which is a system of self-balancing components, is created as a resultant of warping restraint stresses:

$$BM = \int_A \sigma_w \,\hat{\omega}\, dA \tag{4}$$

which can be written as:

$$BM = -E f'' I_{\hat{\omega}} \tag{5}$$

in which

$$I_{\hat{\omega}} = \int_A \hat{\omega}^2 dA \tag{6}$$

$I_{\hat{\omega}}$ is the warping moment of inertia.

In dealing with closed sections, the shear deformation in the median surface of the wall will be taken into consideration by the following expression (Dabrowski 1968):

$$\gamma_{s\bar{x}} = \frac{\partial u_w}{\partial s} + \frac{\partial u_s}{\partial \bar{x}} - \frac{T}{G\delta} \tag{7}$$

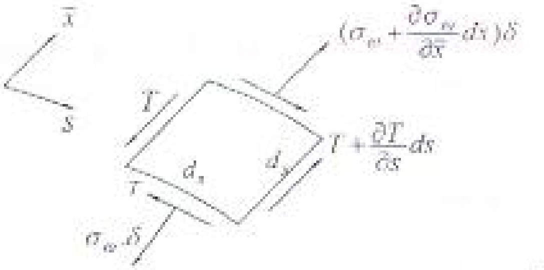


Fig. (2) A differential wall element

The total shear T can be determined from equilibrium conditions for a differential wall element shown in **Fig. (2)**, subjected to loading by normal forces $(\sigma_w\,\delta)$ and shear forces T, as:

$$T = \frac{H}{\Omega} - \frac{BM'}{I_{\hat{\omega}}}\left(S_{\hat{\omega}} - \frac{1}{\Omega}\int_A S_{\hat{\omega}}\, h\, ds\right) \tag{8}$$

This expression of the shear flow consists of two parts, a constant part known as the primary shear flow, which occurs only in the closed part of the section, and the secondary shear flow.

Where:

$$S_{\hat{\omega}} = \int_0^s \hat{\omega}\,\delta\, ds = \int_A \hat{\omega}\, dA \tag{9}$$

$$\overline{S}_{\hat{\omega}} = S_{\hat{\omega}} - \frac{1}{\Omega}\int S_{\hat{\omega}}\, h\, ds \tag{10}$$

The relationship between the functions f and τ may be determined from another description for the total shear T which can be obtained from the equation of shear strain $\gamma_{s\overline{x}}$ Eq. (7):

$$T = G\,\delta(-f'\frac{\partial\hat{\omega}}{\partial s} + h\tau') \tag{11}$$

$$\tau' = \mu\, f' + \frac{H}{GI_c} \tag{12}$$

or

$$\tau' = f' - \frac{E I_{\hat{\omega}}}{G\,\mu\, I_c} f''' \tag{13}$$

Integrating this equation with respect to $\overline{x}$ results:

$$\tau = f - \frac{E I_{\hat{\omega}}}{G\,\mu\, I_c} f'' + C \tag{14}$$

where:

C : is the constant of integration.

μ : is the warping shear parameter, which determined by the following expression.

$$\mu = 1 - \frac{J}{I_c} \tag{15}$$

$$Ic = \int h^2 \delta ds \tag{16}$$

$$J = \frac{\Omega^2}{\oint ds/\delta} \tag{17}$$

FORMULATION OF A CURVED BEAM ELEMENT WITH WARPING

The basic assumptions utilized in the formulation of the curved beam element are (Yoo 1987), (Dabrowski 1968):

1- The element is prismatic.
2- The loads are applied statically and constitute a conservative force system.
3- The cross-section maintains its original shape.
4- The deformations are small with respect to the dimensions of the cross-section (linearized problem).
5- The material is homogeneous, isotropic and obeys Hook's law.
6- The cross-sectional dimensions are small relative to the radius of curvature.
7- The cross-sectional shape is preserved under all loads, or thin-walled beams of a non-deformable cross-section.
8- The projection of the cross-section on a plane normal to the tangential axis does not distort during deformation.

The three dimensional formulation of curved beam elements can be divided into two separate groups:

1- Group-1: an in-plane group considering only displacement in the $\overline{x}, \overline{y}$ plane as an arch.
2- Group-2: a normal-to-plane group considering only displacements in the $\overline{x}, \overline{z}$ plane as a horizontally curved beam.

where $x, y, \bar{z}$ are the local curvilinear coordinates of the curved beam element as shown in **Fig. (3)**. The forces and displacements are positive if their vectors point in the direction of the positive coordinates. Right hand rule is used for the sign of moments and rotations.

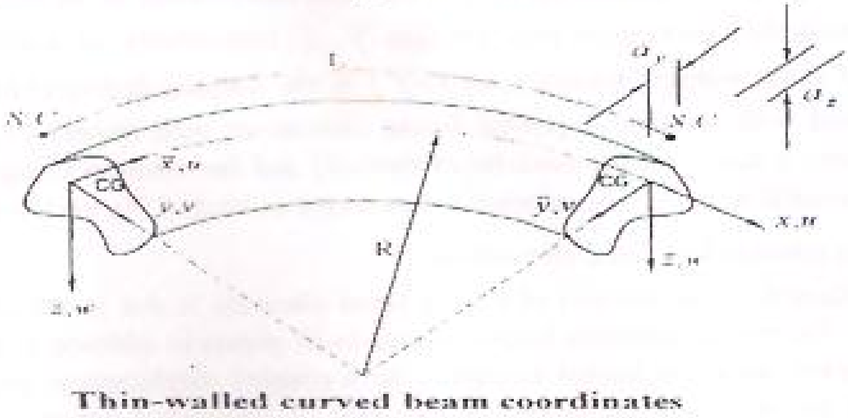
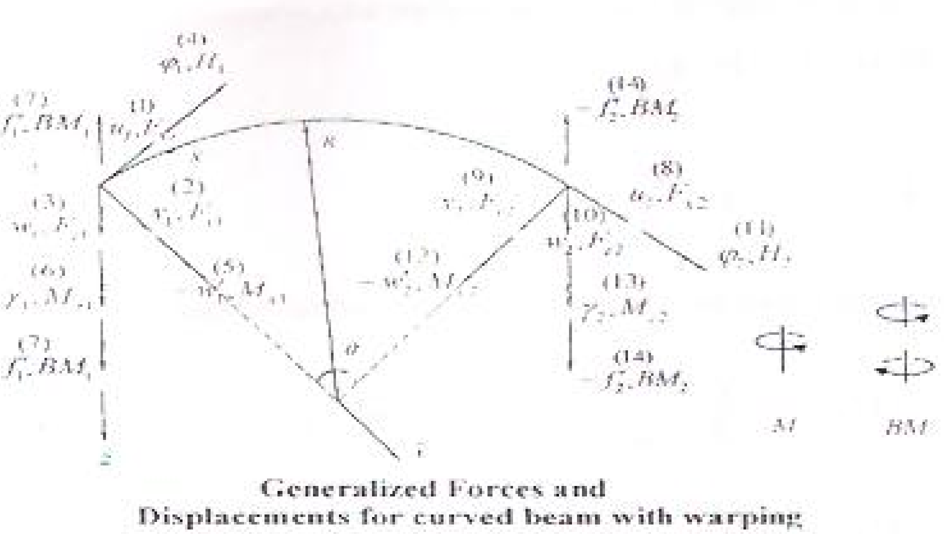


Fig. (3) Thin-walled curved beam element in local curvilinear coordinates

Fig. (4) shows the generalized forces and displacements for the curved beam element with warping.



The relationship between internal forces and the displacements at a point on the middle surface of the member are mentioned in (Dabrowski 1968), (Yoo 1987 & 1979):

$$\left.\begin{aligned}
F_{\bar{x}} &= EA(u' - v/R) \\
M_{\bar{z}} &= EI_{\bar{z}}(v'' + v/R^2) \\
M_{\bar{y}} &= -EI_{\bar{y}}(w'' - \varphi/R) \\
M_{s.v.} &= GJ(\varphi' + w'/R) \\
M_w &= -EI_{\bar{\omega}} f''' \\
H &= M_{s.v.} + M_w \\
BM &= -EI_{\bar{\omega}} f'' \\
F_{\bar{y}} &= -M'_{\bar{z}} \\
F_{\bar{z}} &= M'_{\bar{y}} + H/R
\end{aligned}\right\} \qquad (18)$$

in which $F_{\bar{x}}$ is the axial force; $M_{\bar{y}}$ and $M_{\bar{z}}$ are the bending moments about $\bar{y}$ and $\bar{z}$-axes (using the right-hand rule); H is the total torque, being a vector sum of St.Venant torque $M_{S.V.}$ and warping torque M_w; BM is the bimoment; u is the axial deformation in the direction of $\bar{x}$; v, w are displacements in the directions of principal axes $\bar{y}$, $\bar{z}$, respectively; φ is the angle of rotation about $\bar{x}$-axis; f is the warping function and $(-f')$ is the warping degree-of-freedom for closed sections associated with bimoment. primes denote derivatives with respect to $\bar{x}$-axis; R is the radius of curvature; E and G are the modulus of elasticity and the modulus of rigidity respectively; A is the cross-sectional area; I_y, I_z are principal moments of inertia; J is the torsional constant and $I_{\bar{\omega}}$ is the warping constant for closed thin-walled.

The primary difficulty in the analysis of curved beam elements is due to the coupling action of bending, torsion (in curved members loaded normal-to its plane) in addition to bending and axial extension (in curved members loaded in plane). So a coupled displacement field is derived and developed using the equations of equilibrium in terms of component deformation for the linear static problem projected on coordinate axes.

Equilibrium Equations of Curved Beams

The equilibrium equations of a curved beam loaded in-plane are:-

$$-\frac{EA}{R}(u' - \frac{v}{R}) + EI_{\bar{z}}(v^{iv} + \frac{v''}{R^2}) = -q_{\bar{y}} + m'_{\bar{z}} \qquad (19\text{-a})$$

$$\frac{EI_{\bar{z}}}{R}(v''' + \frac{v'}{R^2}) + EA(u'' - \frac{v'}{R}) = q_{\bar{x}} + \frac{m_{\bar{z}}}{R} \qquad (19\text{-b})$$

While the equilibrium equations for a curved beam loaded normal-to-plane while warping effects are considered are:

$$-EI_{\bar{y}}(w^{iv} + \frac{w''}{R^2}) + (EI_{\bar{y}} + GJ)\frac{f''}{R} - \frac{EI_{\bar{\omega}}}{\mu R}(1 + \frac{EI_{\bar{y}}}{GI_c})f^{iv} = m'_{\bar{y}} + q_{\bar{z}} \qquad (20\text{-a})$$

and

$$-\frac{EI_{\bar{y}}}{R}(w'' - \frac{w}{R^2}) + \frac{EI_{\bar{y}}}{R^2}f - \frac{E^2 I_{\bar{\omega}} I_{\bar{y}}}{G\mu I_c R^2}f'' - GJf'' + \frac{EI_{\bar{\omega}}}{\mu}f^{iv} = -m_{\bar{x}} - \frac{EI_{\bar{y}}C}{R^2} \qquad (20\text{-b})$$

Some forces in eqs. (18), (19) and (20) refer to the line of centroids others refer to shear center. The shear center with coordinates $a_{\bar{y}}$ and $a_{\bar{z}}$, measured from the centroid, is called the principal sectorial pole, or simply, principal pole. Using the same radius of curvature, R, in these equations for gravity center and shear center can be acceptable since in most practical beams the quantity $a_{\bar{y}}/R$ as compared to unity can be neglected without inducing a significant error. Thus the derived displacement field does not have to be limited to curved beams of doubly or singly symmetric sections with the axis of symmetry being in the plane of bending.

General Displacement Fields and Displacement Functions

The generalized displacements and force vectors for a curved beam, as shown in **Fig. (4)**, are:

$$\{\Delta_w\} = [u_1,\ v_1\ \ w_1\ \ \varphi_1\ \ -w'_1\ \ \gamma_1\ \ -f'_1\ \ u_2\ \ v_2\ \ w_2\ \ \varphi_2\ \ -w'_2\ \ \gamma_2\ \ -f'_2]^T \qquad (21)$$

and

$$\{F_w\}=\left[F_{\bar{x}1}\ \ F_{\bar{y}1}\ \ F_{\bar{z}1}\ \ H_1\ \ M_{\bar{y}1}\ \ M_{\bar{z}1}\ \ BM_1\ \ F_{\bar{x}2}\ \ F_{\bar{y}2}\ \ F_{\bar{z}2}\ \ H_2\ \ M_{\bar{y}2}\ \ M_{\bar{z}2}\ \ BM_2\right]^T \tag{22}$$

The subscript w refers to a curved beam with warping. While the subscripts (1) and (2) refer to nodes 1 and 2, respectively, which are the end nodes of the element. The positive direction of the generalized displacements and forces point to the positive local coordinate system as shown in **Fig. (4)**.

A coupled boundary condition is introduced for the first group, curved beam loaded in plane, which is defined in references (Yoo 1979) and (Yoo 1987) as:

$$\gamma = v' + \frac{u}{R} \tag{23}$$

The subdivided generalized displacement vectors are:

$$\{\Delta_{w1}\}=[u_1\ \ v_1\ \ 0\ \ 0\ \ 0\ \ \gamma_1\ \ 0\ \ u_2\ \ v_2\ \ 0\ \ 0\ \ 0\ \ \gamma_2\ \ 0]^T \tag{24-a}$$

$$\{\Delta_{w2}\}=[0\ \ 0\ \ w_1\ \ \varphi_1\ \ -w'_1\ \ 0\ \ -f'_1\ \ 0\ \ 0\ \ w_2\ \ \varphi_2\ \ -w'_2\ \ 0\ \ -f'_2]^T \tag{24-b}$$

$\{\Delta_{w1}\}$ and $\{\Delta_{w2}\}$ are for group-1 and 2 respectively. Also, the subdivided generalized force vectors are:

$$\{F_{w1}\}=\left[F_{\bar{x}1}\ \ F_{\bar{y}1}\ \ 0\ \ 0\ \ 0\ \ M_{\bar{z}1}\ \ 0\ \ F_{\bar{x}2}\ \ F_{\bar{y}2}\ \ 0\ \ 0\ \ 0\ \ M_{\bar{z}2}\ \ 0\right]^T \tag{25-a}$$

and

$$\{F_{w2}\}=\left[0\ \ 0\ \ F_{\bar{z}1}\ \ H_1\ \ M_{\bar{y}1}\ \ 0\ \ BM_1\ \ 0\ \ 0\ \ F_{\bar{z}2}\ \ H_2\ \ M_{\bar{y}2}\ \ 0\ \ BM_2\right]^T \tag{25-b}$$

$\{F_{w1}\}$ and $\{F_{w2}\}$ are for group-1 and 2 respectively.

In-Plane Displacement Field and Displacement Functions

For group-1, the homogeneous case of equilibrium eqs. (19-a, b) can be rewritten as:

$$L_1 v + L_2 u = 0 \tag{26-a}$$

$$L_3 v + L_4 u = 0 \tag{26-b}$$

where:

$$L_1 = -I_z\frac{d^4}{d\bar{x}^4} - \frac{I_z}{R^2}\frac{d^2}{d\bar{x}^2} - \frac{A}{R^2} \tag{27-a}$$

$$L_2 = \frac{A}{R}\frac{d}{d\bar{x}} \tag{27-b}$$

$$L_3 = \frac{I_z}{R}\frac{d^3}{d\bar{x}^3} + \left(\frac{I_{\bar{z}}}{R^3} - \frac{A}{R}\right)\frac{d}{d\bar{x}} \tag{27-c}$$

$$L_4 = A\frac{d^2}{d\bar{x}^2} \tag{27-d}$$

Since the operators are linear and are commutative, the following equation can be applied to v or u:

$$(L_1 L_4 - L_3 L_2) = 0 \tag{28}$$

or

$$\frac{d^6}{d\bar{x}^6} + \frac{2}{R^2}\frac{d^4}{d\bar{x}^4} + \frac{1}{R^4}\frac{d^2}{d\bar{x}^2} = 0 \tag{29}$$

The coupled displacement field of the curved beam element loaded in plane can be obtained from the solution of these differential equations as:

$$v(\bar{x}) = A_1 + A_2 \cos\frac{\bar{x}}{R} + A_6 \sin\frac{\bar{x}}{R} + A_7 \frac{\bar{x}}{R}\cos\frac{\bar{x}}{R} + A_8 \frac{\bar{x}}{R}\sin\frac{\bar{x}}{R} \tag{30-a}$$

$$u(\bar{x}) = A_1 \frac{\bar{x}}{R} + A_2 \sin\frac{\bar{x}}{R} - A_6 \cos\frac{\bar{x}}{R} + A_7 [F_A \cos\frac{\bar{x}}{R} + \frac{\bar{x}}{R}\sin\frac{\bar{x}}{R}] + A_8 [F_A \sin\frac{\bar{x}}{R} - \frac{\bar{x}}{R}\cos\frac{\bar{x}}{R}] + A_{12} \tag{30-b}$$

$\gamma(\bar{x})$ can be obtained from eq. (23)

$$\gamma(\bar{x}) = A_1 \frac{\bar{x}}{R^2} + A_7 \frac{1+F_A}{R}\cos\frac{\bar{x}}{R} + A_8 \frac{1+F_A}{R}\sin\frac{\bar{x}}{R} + \frac{A_{12}}{R} \tag{30-c}$$

where:

$$F_A = 1 - \frac{2I_{\bar{z}}}{AR^2} \tag{31}$$

which may be called the Winkler's constant (Yoo 1979) and (Yoo 1987).

Normal-to-Plane Displacement Field and Displacement Functions With Warping

A coupled model of a curved beam element loaded normal-to-plane with warping, as an explicit degree-of-freedom, is derived. In addition to the coupling action between bending and torsion, this model considers the total shear deformation (primary and secondary shear deformation), which is caused by warping; the coupled displacement field is derived from the homogeneous case of equilibrium equations (20-a, b):

$$L_5 w + L_6 f = 0 \tag{32-a}$$

$$L_7 w + L_8 f = 0 \tag{32-b}$$

where:

$$L_5 = -EI_{\bar{y}} \frac{d^4}{d\bar{x}^4} - \frac{EI_{\bar{y}}}{R^2}\frac{d^2}{d\bar{x}^2} \tag{33-a}$$

$$L_6 = -\frac{EI_{\hat{\omega}}}{\mu R}(1+\frac{EI_{\bar{y}}}{GI_c})\frac{d^4}{d\bar{x}^4} + \frac{(EI_{\bar{y}}+GJ)}{R}\frac{d^2}{d\bar{x}^2} \tag{33-b}$$

$$L_7 = -\frac{EI_{\bar{y}}}{R}\frac{d^2}{d\bar{x}^2} - \frac{EI_{\bar{y}}}{R^3} \tag{33-c}$$

$$L_8 = \frac{EI_{\hat{\omega}}}{\mu}\frac{d^4}{d\bar{x}^4} - (\frac{E^2 I_{\hat{\omega}} I_{\bar{y}}}{GR^2 \mu I_c} + GJ)\frac{d^2}{d\bar{x}^2} + \frac{EI_{\bar{y}}}{R^2} \tag{33-d}$$

Also these operators are linear and are commutative, therefore the following equation can be applied to w or f:

$$(L_5 L_8 - L_6 L_7) = 0 \tag{34}$$

or

$$R^2 \frac{d^8}{d\bar{x}^8} + (2 - \frac{R^2 \mu GJ}{EI_{\hat{\omega}}})\frac{d^6}{d\bar{x}^6} + (\frac{1}{R^2} - \frac{2\mu GJ}{EI_{\hat{\omega}}})\frac{d^4}{d\bar{x}^4} - \frac{\mu GJ}{R^2 EI_{\hat{\omega}}}\frac{d^2}{d\bar{x}^2} = 0 \tag{35}$$

Solving these differential equations results the coupled displacement field of the curved beam element loaded normal-to-plane while warping effects are considered:

$$w(\bar{x}) = A_3 + A_4\frac{\bar{x}}{R} + A_5\cosh\frac{\hat{\alpha}\,\bar{x}}{R} + A_7\sinh\frac{\hat{\alpha}\,\bar{x}}{R} + A_{10}\cos\frac{\bar{x}}{R} + A_{11}\sin\frac{\bar{x}}{R} + A_{12}\frac{\bar{x}}{R}\cos\frac{\bar{x}}{R} + A_{14}\frac{\bar{x}}{R}\sin\frac{\bar{x}}{R} \tag{36-a}$$

and

$$f(\bar{x}) = \frac{A_3}{R} + A_4\frac{\bar{x}}{R^2} + A_5\frac{(1+\hat{\alpha}^2)}{R\mu}\cosh\frac{\hat{\alpha}\,\bar{x}}{R} + A_7\frac{(1+\hat{\alpha}^2)}{R\mu}\sinh\frac{\hat{\alpha}\,\bar{x}}{R} - A_{12}\frac{ABIC}{R}\sin\frac{\bar{x}}{R} + A_{14}\frac{ABIC}{R}\cos\frac{\bar{x}}{R} \tag{36-b}$$

while φ can be obtained from this equation $\varphi = -\frac{w}{R} + f - \frac{EI_{\hat{\omega}}}{G\,\mu\,I_c}f'' + C$

$$\varphi(\bar{x}) = A_5\frac{\hat{\alpha}^2}{R}\cosh\frac{\hat{\alpha}\,\bar{x}}{R} + A_7\frac{\hat{\alpha}^2}{R}\sinh\frac{\hat{\alpha}\,\bar{x}}{R} - \frac{A_{10}}{R}\cos\frac{\bar{x}}{R} - \frac{A_{11}}{R}\sin\frac{\bar{x}}{R} - \frac{A_{12}}{R}\left[ABICW\sin\frac{\bar{x}}{R} + \frac{\bar{x}}{R}\cos\frac{\bar{x}}{R}\right] + \frac{A_{14}}{R}\left[ABICW\cos\frac{\bar{x}}{R} - \frac{\bar{x}}{R}\sin\frac{\bar{x}}{R}\right] \tag{36-c}$$

in which:

$$ABIC = \frac{2\hat{\beta}^2}{1+\hat{\alpha}^2+\hat{\beta}^2+\hat{\xi}^2} \tag{37}$$

$$ABICW = \frac{2(\hat{\beta}^2+\hat{\xi}^2)}{1+\hat{\alpha}^2+\hat{\beta}^2+\hat{\xi}^2} \tag{38}$$

$$\hat{\alpha} = \sqrt{R^2\frac{\mu\,GJ}{EI_{\hat{\omega}}}} \tag{39}$$

$$\hat{\beta} = \sqrt{R^2\frac{\mu\,I_{\bar{y}}}{I_{\hat{\omega}}}} \tag{40}$$

and

$$\hat{\xi} = \sqrt{\frac{EI_{\bar{y}}}{GI_c}} \tag{41}$$

Strain Energy of The Curved Beam Element

In addition to the normal stresses due to bending and axial force, which are developed in a curved beam, an additional axial direct stress (or normal warping stress) and both primary and secondary shear stresses due to the total torque and the bimoment, respectively will be formed in the horizontally curved beam when restrained warping effects are included. As the cross-sectional dimensions are small compared to the radius of curvature, the shear stresses due to bending can be neglected.

The relationship between forces and normal or shear stresses of thin-walled curved beam elements are:

$$\sigma_{\bar{x}} = \sigma_{\bar{x}1} + \sigma_{\bar{x}2} \tag{42}$$

in which $\sigma_{\bar{x}1}$ and $\sigma_{\bar{x}2}$ are the normal stresses of groups-1 and 2, respectively.

$$\sigma_{x1} = \frac{F_x}{A} - \frac{M_{\bar{z}}.\bar{y}}{I_{\bar{z}}} \tag{43}$$

$$\sigma_{x2} = \frac{M_{\bar{y}}\,\bar{z}}{I_{\bar{y}}} + \frac{BM\,\hat{\omega}}{I_{\hat{\omega}}} \tag{44}$$

$$\tau_{sx} = G[\frac{\Omega\tau'}{\oint ds} + (\frac{\Omega}{\oint ds} - h)(f' - \tau')] \tag{45}$$

The strain energy is:

$$U = U_1 + U_2 \tag{46}$$

where U_1 and U_2 are strain energies of groups-1 and 2, respectively.

$$U_1 = \frac{E}{2}\int_{\bar{x}}[A(u' - \frac{v}{R})^2 + I_{\bar{z}}(v'' + \frac{v}{R^2})^2]d\bar{x} \tag{47}$$

The strain energy of group-2 comprises strain energies due to normal and shears deformations

$$U_2 = U_{2n} + U_{2s} \tag{48}$$

The strain energy due to normal stress for group-2 is:

$$U_{2n} = \frac{E}{2}\int_{\bar{x}}[I_{\bar{y}}(w'' - \frac{\varphi}{R})^2 + I_{\hat{\omega}}f''^2]d\bar{x} \tag{49}$$

While the strain energy due to the primary and secondary shear deformations that will be appeared when restrained warping effects are included, U_{2s}, can be expressed as:

$$U_{2s} = \frac{GJ}{2}\int_{\bar{x}}\tau'^2 d\bar{x} + \frac{G}{2}\int_{\bar{x}}(I_c - J)(f' - \tau')^2 d\bar{x} \tag{50}$$

Stiffness Matrix of Curved Beams With Warping

A stiffness of order (14x14) is derived by minimizing the potential energy. Due to the complication of [B] matrix expressions, the numerical integration technique is used to produce the stiffness matrix elements.

$$[K_L] = [K1] + [K2] \tag{51}$$

in which

[K_L]: is the total stiffness matrix for the curved beam element with warping in the local coordinate.

[$K1$]: is the stiffness matrix of order (6x6) for the curved beam loaded in plane (group-1)

[K2]: is the stiffness matrix of order (8x8) for the curved beam loaded normal-to-plane (group-2) when warping effects being included.

In-Plane Stiffness Matrix

The stiffness matrix of this group consists of axial and bending stiffness matrices, as:

$$[K1] = [KA] + [KB1] \tag{52}$$

in which [$KB1$] denotes bending stiffness matrix for group-1:

$$[KA]=EA\int_0^L [BA]^T.[BA]d\bar{x} \tag{53-a}$$

and

$$[KBI]=EI_z\int_0^L [BBI]^T.[BBI]d\bar{x} \tag{53-b}$$

in which $[BA]$ and $[BBI]$ are row vectors relating the joint displacements to strain field for group-1 ($\varepsilon=[B]\{\Delta\}$)

$$[BA]=[N_u'-N_v/R] \tag{54-a}$$

and

$$[BBI]=[N_v''+N_v/R^2] \tag{54-b}$$

The individual element for $[KA]$ and $[KBI]$ can be expressed as:

$$KA_{ij}=EA\int_0^L(N_u'-\frac{N_v}{R})i.(N_u'-\frac{N_v}{R})j\,d\bar{x} \tag{55-a}$$

and

$$KB1_{ij}=EI_z\int_0^L(N_v''+\frac{N_v}{R^2})i.(N_v''+\frac{N_v}{R^2})j\,d\bar{x} \tag{55-b}$$

where i or j = (1, 2, 6, 8, 9, 13)

$$(N_u'-\frac{N_v}{R})i=A_{(7,i)}\frac{(1-F_A)}{R}sin\frac{\bar{x}}{R}-A_{(8,i)}\frac{(1-F_A)}{R}cos\frac{\bar{x}}{R} \tag{56-a}$$

$$(N_v''+\frac{N_v}{R^2})i=\frac{A_{(1,i)}}{R^2}-A_{(7,i)}\frac{2}{R^2}sin\frac{\bar{x}}{R}+A_{(8,i)}\frac{2}{R^2}cos\frac{\bar{x}}{R} \tag{56-b}$$

Normal-to-Plane Stiffness Matrix

The stiffness matrix of this group comprises of four subdivided stiffness matrices, as in the following:

$$[K2]=[KB2]+[KBM]+[KT1]+[KT2] \tag{57}$$

where:

$[KB2]$: is the bending stiffness matrix for group-2 when warping is considered.

$[KBM]$: is the stiffness matrix resulting from the warping normal stress and strain caused by the bimoment.

$[KT1]$ and $[KT2]$: are torsional stiffness matrices resulting from the total shear deformation, which comprises the primary and secondary shear deformations.

$$[KB2]=EI_y\int_0^L [BB2]^T[BB2]d\bar{x} \tag{58-a}$$

$$[KBM]=EI_\omega\int_0^L [BBM]^T[BBM]d\bar{x} \tag{58-b}$$

$$[KT1]=GJ\int_0^L [BT1]^T[BT1]d\bar{x} \tag{58-c}$$

$$[KT2]=G(I_c-J)\int_0^L [BT2]^T[BT2]d\bar{x} \tag{58-d}$$

in which $[BB]$, $[BBM]$, $[BT1]$, and $[BT2]$ are row vectors relating the joint degrees-of-freedom to strain field for group-2 with warping.

$$[BB2]=\left[N''_w-\frac{N_\varphi}{R}\right] \tag{59-a}$$

$$[BBM]=[N''_f] \tag{59-b}$$

$$[BT1]=[N'_\tau] \tag{59-c}$$

$$[BT2]=[N'_f-N'_\tau] \tag{59-d}$$

in which

$$N_\tau=N_\varphi+\frac{N_w}{R} \tag{60}$$

The expressions for the individual element of the subdivided stiffness matrices are:

$$KB2_{ij}=E\,I_{\bar{y}}\int_0^L\left(N''_w-\frac{N_\varphi}{R}\right)i\left(N''_w-\frac{N_\varphi}{R}\right)j\,dx \tag{61-a}$$

$$KBM_{ij}=E\,I_{\bar{\omega}}\int_0^L(N''_f)i(N''_f)j\,d\bar{x} \tag{61-b}$$

$$KT1_{ij}=GJ\int_0^L(N'_\tau)i(N'_\tau)j\,d\bar{x} \tag{61-c}$$

$$KT2_{ij}=G(I_c-J)\int_0^L(N'_f-N'_\tau)i(N'_f-N'_\tau)j\,d\bar{x} \tag{61-d}$$

where i, j = (3, 4, 5, 7, 10, 11, 12, 14).

The individual elements of the $[B]$ vectors in eqs. (59) are:

$$\left(N''_w-\frac{N_\varphi}{R}\right)_i=\frac{A_{(12,i)}}{R^2}(ABICW-2)\sin\frac{\bar{x}}{R}-\frac{A_{(14,i)}}{R^2}(ABICW-2)\cos\frac{\bar{x}}{R} \tag{62-a}$$

$$(N''_f)_i=\frac{A_{(5,i)}}{R^3\mu}\hat{\alpha}^2(1+\hat{\alpha}^2)\cosh\frac{\hat{\alpha}\,\bar{x}}{R}+\frac{A_{(7,i)}}{R^3\mu}\hat{\alpha}^2(1+\hat{\alpha}^2)\sinh\frac{\hat{\alpha}\,\bar{x}}{R}+\frac{A_{(12,i)}}{R^3}ABIC\sin\frac{\bar{x}}{R}-\frac{A_{(14,i)}}{R^3}ABIC\cos\frac{\bar{x}}{R} \tag{62-b}$$

$$(N'_\tau)_i=\frac{A_{(4,i)}}{R^2}+\frac{A_{(5,i)}}{R^2}\hat{\alpha}(1+\hat{\alpha}^2)\sinh\frac{\hat{\alpha}\,\bar{x}}{R}+\frac{A_{(7,i)}}{R^2}\hat{\alpha}(1+\hat{\alpha}^2)\cosh\frac{\hat{\alpha}\,\bar{x}}{R}-\frac{A_{(12,i)}}{R^2}ABICW\cos\frac{\bar{x}}{R}-\frac{A_{(14,i)}}{R^2}ABICW\sin\frac{\bar{x}}{R} \tag{62-c}$$

$$(N'_f-N'_\tau)_i=A_{(5,i)}\frac{J}{R^2\mu I_c}\hat{\alpha}(1+\hat{\alpha}^2)\sinh\frac{\hat{\alpha}\,\bar{x}}{R}+A_{(7,i)}\frac{J}{R^2\mu I_c}\hat{\alpha}(1+\hat{\alpha}^2)\cosh\frac{\hat{\alpha}\,\bar{x}}{R}+A_{(12,i)}\frac{EI_{\bar{\omega}}}{R^4G\mu I_c}ABIC\cos\frac{\bar{x}}{R}++A_{(14,i)}\frac{EI_{\bar{\omega}}}{R^4G\mu I_c}ABIC\sin\frac{\bar{x}}{R} \tag{62-d}$$

TRANSFORMATION INTO SYSTEM CORDINATES

The stiffness matrices of the three-dimensional curved beam elements refer to the element local axes $\bar{x}, \bar{y}, \bar{z}$. The total transformation of the stiffness matrices of curved beam elements from local coordinates can be expressed as:

$$[K_G]=[T]^T[T']^T[K_L][T'][T] \tag{63}$$

where

$[K_G]$: is the global stiffness matrix of a curved beam element.

$[K_L]$: is the stiffness matrix of curved beam element in the local coordinate system.

On the other hand, $[T']$ is the transformation matrix used for the first stage of transformation from the local curvilinear coordinates $\bar{x}, \bar{y}, \bar{z}$ to the local straight coordinates x, y, z **(Fig. (5).** The elements of this matrix are the direction cosines between the two systems ($\bar{x}, \bar{y}, \bar{z}$) and ($x$, y, z). For curved beams with warping, the $[T']$ matrix will be of order of (14x14), which can be expressed as:

$$[T'] = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & c & -s & 0 & & & & & & \text{SYMM.} \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{64}$$

where $s = sin(\theta / 2)$, and $c = cos(\theta / 2)$.

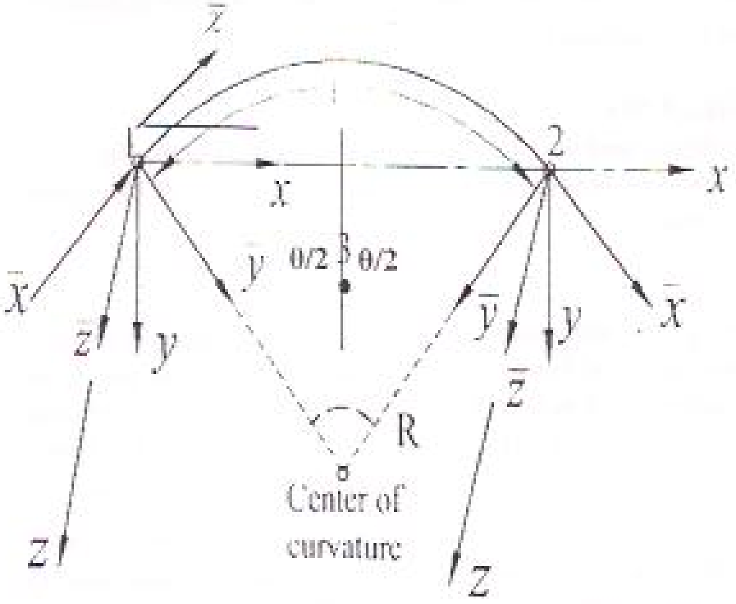


Fig. (5) Curvilinear coordinates $\bar{x}, \bar{y}, \bar{z}$ and straight coordinates x, y, z

$[T]$ Matrix in Eq. (63) is for the second stage of transformation to convert the stiffness matrix from local straight coordinates to the global coordinate system. For curved beams with warping, $[T]$ can be expressed as:

$$[T]=\begin{bmatrix} [T''] & 0 & 0 & 0 & 0 & 0 \\ & [T''] & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & [T''] & 0 & 0 \\ & \text{SYMM.} & & & [T''] & 0 \\ & & & & & 1 \end{bmatrix} \tag{65}$$

where $[T'']$ is the (3x3) transformation matrix as described by reference (Dabrowski 1968)and can be expressed as (Azar 1972 & Krishnamoorthy 1988):

$$[T'']=\begin{bmatrix} \cos\alpha.\cos\beta & \sin\beta & \sin\alpha.\cos\beta \\ -\cos\alpha.\sin\beta.\cos\gamma \\ -\sin\alpha.\sin\gamma & \cos\gamma.\cos\beta & -\sin\alpha.\sin\beta.\cos\gamma \\ & & +\cos\alpha.\sin\gamma \\ \cos\alpha.\sin\beta.\sin\gamma & -\sin\gamma.\cos\beta & \sin\alpha.\sin\beta.\sin\gamma \\ -\sin\alpha.\cos\gamma & & +\cos\alpha.\cos\gamma \end{bmatrix} \tag{66}$$

where α, β, γ are rotations about Y, Z, X, respectively.

The value (1) in $[T]$and $[T']$, in rows 7 and 14 is the direction cosines of the member for the seventh degree-of -freedom (warping).

VERIFICATION PROBLEMS

A computer program is developed for static analysis to demonstrate the accuracy of the developed curved beam element. Two problems are analyzed. The results of the present study analysis are compared with exact solutions.

Problem (1)

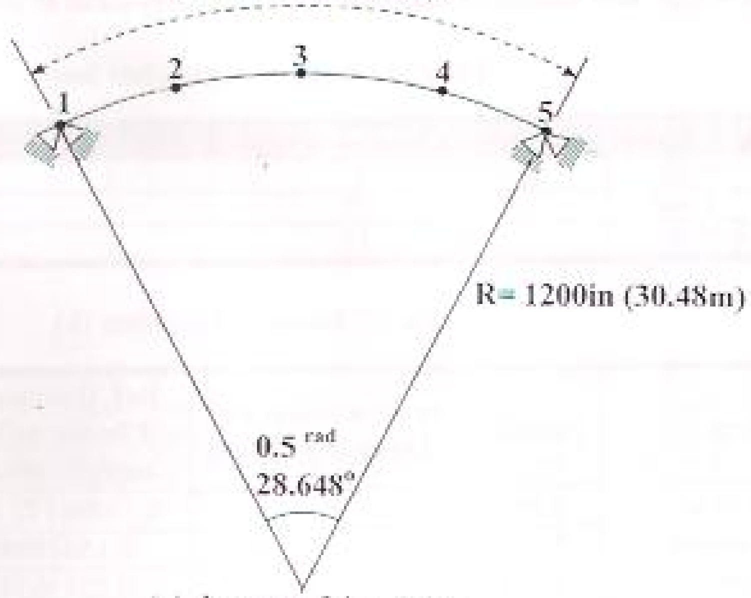
A single span curved box girder subjected to truck eccentric loading is analyzed. Effects of warping are considered in the analysis. The loading and geometry of the box girder of this problem with its sectional properties are shown in **Table. (1)** and **Fig. (6)**. The modulus of elasticity E and Poisson's ratio υ are 30000 k / in^2 (206832.41 MPa) and (0.2931), respectively. Both supports of the girder are pinned in bending and torsion $(M_y = 0,\ \varphi = 0,\ BM = 0)$. Four elements are used in the analysis of this girder. **Table. (2)** shows a comparison between the present study analysis results, when curved beam elements with warping are used and reference (Heins and Oleinik 1976) analysis results. This reference analyzed the same problem and presented results where 200 curved beam elements that he developed were used. Also, this comparison includes the results of the closed-form solution from reference (Dabrowski 1968). The closed-form solution equations are presented in Appendix (A). The results of the present study analysis show an excellent agreement with the closed-form solution results and the maximum difference is ($3.232\text{x}10^{-5}$%) from these results. On the other hand, the present study analysis results show a maximum difference of (.174 %) from the displacements and rotations results of reference (Heins and Oleinik 1976). The values of the bimoment in reference (Heins and Oleinik 1976) are found to be far from the closed form solution values, while the present analysis bimoment values are very close to those of the exact solution.

Table.(1) Problem (1) applied loads

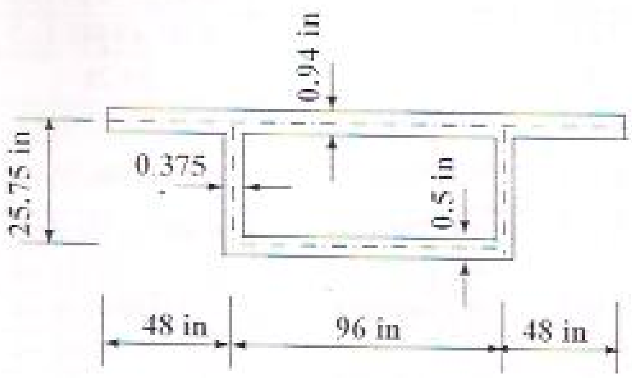
Node	Location	Vertical concentrated load (k)	Concentrated torsion (k.in)
2	0.22L	6	-288
3	0.50L	32	-1152
4	0.78L	32	-1152

Table.(2) Results of problem (1)

Item	Location	Exact solution Ref. (Dabrowski 1968)	Ref. (Heins and Oleinik 1976) analysis results	Present study results
Vertical displacement w (in)	0.05L		0.53808172 E-1	0.53804271 E-1
	0.15L		0.15688980	0.15687833
	0.22L		0.22150741	0.22149105
Rotation φ (rad)	0.05L		-0.14346389 E-3	-0.14332976 E-3
	0.15L		-0.42225950 E-3	-0.42172726 E-3
	0.22L		-0.60145488 E-3	-0.60250367 E-3
Shear force $F_{\bar{z}}$ (k)	0		29.28	29.28
Bending moment M_y (k.in)	0.05L	0.93261162E+3	0.93261258E+3	0.93261151 E+3
	0.15L	0.27955038E+4	0.27955067E+4	0.27955037 E+4
	0.22L	0.40956474E+4	0.40956516E+4	0.40956472 E+4
Total torque H (k.in)	0	-0.21723501E+4	-0.21725838E+4	-0.21723502E+4
	0.05L	-0.21606918E+4	-0.21606924E+4	-0.21606920E+4
	0.15L	-0.20674695E+4	-0.20674700E+4	-0.20674697E+4
T_{sv} (k.in)	0	-0.21653588E+4	-0.21680428E+4	-0.21653590E+4
	0.05L	-0.21532267E+4	-0.21563426E+4	-0.21532268E+4
	0.15L	-0.20536354E+4	-0.20576431E+4	-0.20536356E+4
T_w (k.in)	0	-0.69912562E+1	-0.43073081E+1	-0.69912557E+1
	0.05L	-0.74651696E+1	-0.43497252E+1	-0.74651694E+1
	0.15L	-0.13834114E+2	-0.98269119E+1	-0.13834115E+2
BM (k.in^2)	0.05L	-0.21436112E+3	-0.12958343E+3	-0.21436111E+3
	0.15L	-0.79732544E+3	-0.45730444E+3	-0.79732543E+3
	0.22L	-0.16957586E+4	-0.24010951E+4	-0.16957586E+4



(a): Layout of the girder



(b): Cross section

Section properties:

J = 0.567513128 E+5 in^4 (2.3621679 E -2 m^4) $I_{\bar{y}}$ =0.2724672458 E +5 in^4 (1.1340943 E -2 m^4),

$I_{\bar{\omega}}$ =4.914587775 E +6 in^6 (1.319743 E -3 m^6), I_c = 0.6975613917 E +5 in^4 (2.9034697 E -2 m^4)

Fig. 6 : Curved box girder of problem (1)

Problem (2)

A semicircular fixed ended arch of radius (R=1.5 m) is analyzed. The sectional properties are (J= 6.4E-3, I_z = 1.6E-3 m^4). The modulus of elasticity is (E = 200000 MPa) and Poisson's ratio is (υ=0.3). To assess the validity of both in-plane action of the developed curved beam element; the arch is loaded by a concentrated load of (250 kN) at the crown in the plane of the arch.. The analyzed arch consists of two elements. The results of this analysis are compared with the exact solutions of reference (Martin 1966). The comparison, which is given in **Table (3)**, shows an excellent agreement with the exact solution and the maximum difference is (.187 %).

Table (3) Results of problem (2)

Item	Exact solution results from Reference (Martin 1966)	Present study results
Vertical displacement under the load (cm)	$PR^3/85.802EI_{\bar{z}}=0.3073E\text{-}2$	0.3074890E-2
Bending moment at supports $M_{\bar{z}}$ (kN.cm)	0.1104 PR=0.4140E+4	0.414774E+4
Bending moment under the load (kN.cm)	0.1514PR=0.56775E+4	0.568005E+4

CONCLUSIONS

1- Since the new displacement field is derived from the homogeneous part of the equilibirum equations that control the static behavior of a thin-walled curved beam element of a closed section :when warping effects are included, the element's stiffness matrix that derived using this displacement field gives very closed results compared with the closed form solution results and can be said it is an exact solution therefore there is no added advantage or increase in the accuracy when implying finer mesh.

2- Considering the non-uniform torsional behavior of a closed thin-walled section of the curved beam element in the derivation of its warping function makes the results very close to the closed form solution.

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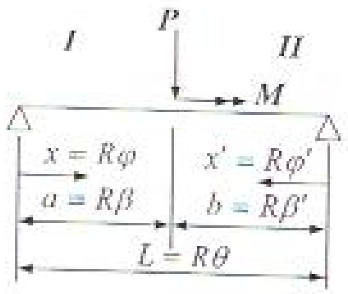
Appendix A

Table (A.1): Stress resultants in the basic system for a loading system comprising a concentrated load ***P*** and a separate twisting moment ***M*** Ref. (Dabrowski 1968).

Quantity	Rang.	Expression
M_y	I	$(PR-M)\frac{\sin\beta'}{\sin\theta}\sin\varphi$
	II	$(PR-M)\frac{\sin\beta}{\sin\theta}\sin\varphi'$
H	I	$(M-PR)\frac{\sin\beta'}{\sin\theta}\cos\varphi+PR\frac{\beta'}{\theta}$
	II	$-(M-PR)\frac{\sin\beta}{\sin\theta}\cos\varphi'-PR\frac{\beta}{\theta}$
$\frac{M_w}{\mu}$	I	$\frac{M(1-\eta)+PR\eta}{k}\frac{\sinh kb}{\sinh kL}\sinh kx+(MR-PR^2)\eta\frac{\sin\beta'}{\sin\theta}\sin\varphi$
	II	$\frac{M(1-\eta)+PR\eta}{k}\frac{\sinh ka}{\sinh kL}\sinh kx'+(MR-PR^2)\eta\frac{\sin\beta}{\sin\theta}\sin\varphi'$
$\frac{BM}{\mu}$	I	$[M(1-\eta)+PR\eta]\frac{\sinh kb}{\sinh kL}\cosh kx+(M-PR)\eta\frac{\sin\beta'}{\sin\theta}\sin\varphi$
	II	$-[M(1-\eta)+PR\eta]\frac{\sinh ka}{\sinh kL}\cosh kx'-(M-PR)\eta\frac{\sin\beta}{\sin\theta}\sin\varphi'$
k		$\sqrt{\frac{\mu GJ}{EI_{\bar{\omega}}}}$
η		$\frac{1}{1+(kR)^2}$



LIST OF SYMBOLS

$a_{\bar{y}}, a_{\bar{z}}$	:	Coordinates of shear center measured from centroid.
A	:	Cross-sectional area.
$ABIC, ABICW$	:	Dimensionless parameters defining cross-sectional rigidities.
A_i	:	Displacement field unknowns.
$[BA]$, $[BBI]$	:	Row vectors relating the joint degree-of-freedom to strain field of group

		–1.
$[BB2]$, $[BT]$	:	Row vectors relating the joint degree-of-freedom to strain field of group –2 without warping.
$[BB_W]$, $[BBM_W]$, $[BT_{W1}]$, $[BT_{W2}]$	:	Row vectors relating the joint degree-of-freedom to strain field of group –2 with warping.
BM	:	Bimoment.
$\{D_I\}$	:	Generalized displacement field for groups –1.
$\{D_{w2}\}$	:	Generalized displacement field for group –2 when warping is considered.
E	:	Modulus of elasticity.
$F_{\bar{x}}, F_{\bar{y}}, F_{\bar{z}}$	:	Axial and shear forces.
f	:	**Dimensionless warping function.**
$\{F_w\}, \{\Delta_w\}$	:	Generalized force and displacement vectors for the curved beam element with warping.
F_A	:	Dimensionless parameters.
G	:	Shear modulus.
h	:	The distance of contour from the shear center.
H	:	Total torque.
$I_{\bar{y}}, I_{\bar{z}}$	:	Principal moments of inertia.
$I_{\bar{\omega}}$	:	Warping moment of inertia for closed and open-closed sections.
I_c	:	Central second moment of area.
J	:	St. Venant's torsional constant.
$[K_1]$	:	Stiffness matrix for a curved beam element loaded in plane.
$[K_2]$	:	Stiffness matrix for a curved beam element normal to plane.
$[K_A]$	:	Axial stiffness matrix of group –1.
$[KB_1]$, $[KB_2]$	:	Bending stiffness matrices of groups –1 and –2.
$[K_{W2}]$	:	Stiffness matrix for a curved beam element normal to plane when warping is considered.
$[KBM_W]$	:	Stiffness matrix results from warping normal stress and strain caused by bimoment.
$[KB_W]$	:	Bending stiffness matrices of group–2 when warping is considered.
$[KT_{W1}]$, $[KT_{W2}]$	:	Torsional stiffness matrices from result primary and secondary shear deformation.
$[K_G]$	:	Global stiffness matrix of curved beam elements.
$[K_L]$,	:	Total stiffness matrix for curved beam elements without warping in the local coordinate.
$[K_{LW}]$	:	Total stiffness matrix for curved beam elements with warping in the local coordinate.
L	:	Element length.
$L_1, L_2, ...L_{11}$	:	Differential operators.
$M_{s.v.}$, M_w	:	St. Venant's and warping torques.
$M_{\bar{y}}, M_{\bar{z}}$	:	Bending moments.
$[N_1]$	:	Shape function matrices for groups –1.
$[N_{w2}]$	:	Shape function matrices for group –2 when warping are considered.
R	:	Radius of curvature.
r	:	The polar radius of gyration.
s	:	The contour ordinate measured from a selected point on the median line of the section.
$S_{\bar{\omega}}$	:	The principal sectorial static moment.

T	:	Shear flow.
$[T]$	:	Transformation matrix.
U	:	Strain energy.
u, v, w	:	Displacements in direction of $\bar{x}, \bar{y}, \bar{z}$.
u_w, u_s	:	Axial and tangential components of the median surface displacement of the plate wall.
$\bar{x}, \bar{y}, \bar{z}$	:	Local curvilinear coordinates.
x, y, z	:	Local straight coordinates.
X, Y, Z	:	Global coordinates.
$\hat{\alpha}$	:	Decay coefficient for non-uniform torsion of closed and open-closed sections.
$\hat{\beta}, \hat{\zeta}$	:	Dimensionless parameters.
γ	:	Kinematic degree-of-freedom.
δ	:	Wall thickness in general.
θ	:	Subtended angle of the curved beam element.
μ	:	Warping shear parameter for closed and open-closed sections.
υ	:	Poisson's ratio.
σ_{x1}, σ_{x2}	:	Normal stresses of groups -1 and -2.
σ_w	:	Warping normal stress.
τ_{sx2}	:	Shear stress of group -2.
τ	:	Total angle twist ($\tau = \varphi + w / R$).
φ	:	Angle of twist.
ω	:	Sectorial area.
$\hat{\omega}$	:	Unit warping for closed and open-closed sections.
Ω	:	Twice the area of closed part of a section enclosed by the median line of the wall.