



A GENERAL VELOCITY PROFILE FOR A LAMINAR BOUNDARY LAYER OVER FLAT PLATE WITH ZERO INCIDENCE

Qussai J. Abdul-Ghafour*

Department of Machines and Equipment, University of Technology, Baghdad, Iraq

ABSTRACT

A general velocity profile for a laminar flow over a flat plate with zero incidence is obtained by employing a new boundary condition to the other available boundary conditions. The general velocity profile is mathematically simple and nearest to the exact solution. Also other related values, boundary layer thickness, displacement thickness, momentum thickness and coefficient of friction are nearest to the exact solution compared with other corresponding values for other researchers.

الخلاصة

تم ايجاد معادلة عامة للسرعة خلال الطبقة المتاخمة للجريان الطبقي فوق السطح المستوي من خلال توظيف ظرف محيطي جديد بالاضافة الى الظروف المحيطية المتوفرة. المعادلة العامة المستحصلة ذات صيغة رياضية بسيطة واقرب ماتكون من الحل التام. تم استخدام المعادلة العامة للسرعة لايجاد القيم ذات العلاقة، سمك الطبقة المتاخمة و سمك الازاحة و سمك الزخم و معامل الاحتكاك وكانت النتائج هي الاقرب للحل التام مقارنة مع القيم المناظرة لها والمستحصلة من الباحثين السابقين.

Keywords: Laminar Boundary Layer, Flat Plate, General Velocity Profile

INTRODUCTION

The boundary layer is a thin layer formed when a real flow passed over a solid surface. The velocity of the flow changes through this layer. At the surface, the velocity of the fluid relative to the surface is zero. The velocity of the flow then increases rapidly from zero and approaches the velocity of the main stream.

In 1904 German engineer Ludwig Prandtl (Schlichting 2000) suggested that the flow may be considered in two parts, the first part is at the boundary layer, where the shear stress is of prime importance, and the second is beyond the boundary layer where the velocity gradient is small and so the effect of viscosity is negligible. In this part the flow is essentially of an ideal fluid.

The boundary layer thickness (δ) is the value of height from the plate surface for which ($u \approx u_\infty$), and the boundary layer velocity profile refers to the manner which (u) varies with (y) through the boundary layer, as shown in **Fig.1** (Massey 2006).

The concept displacement thickness (δ^*) (Massey 2006), allows us to consider the main flow as that of a frictionless fluid past a displaced surface instead of the actual flow past the actual surface. In other words, to reduce the total volume flow rate of a frictionless fluid by the same amount, the surface would have to be displaced outwards a distance (δ^*). **Fig.2** shows observation of (δ^*), where **Fig.2a** shows the corresponded behavior of ideal flow. The shaded areas in **Fig.2** are equal and can be presented mathematically as:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy \tag{1}$$

By similar conception, the total reduction in momentum flow rate equals the momentum flow rate under frictionless condition through thickness (θ) which called Momentum thickness (Massey 2006). Mathematically:

$$\theta = \int_0^\delta \left(\frac{u}{u_\infty}\right) \left(1 - \frac{u}{u_\infty}\right) dy \tag{2}$$

The surface shear stress (τ_w) (Massey 2006) may be evaluated from knowledge of the velocity gradient at the surface that is,

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{3}$$

From external flow, a dimensionless parameter from which the surface frictional drag may be determined, called skin friction coefficient (Massey 2006). In mathematical form,

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho u_\infty^2} \tag{4}$$

The momentum integral equation of the boundary layer, first derived by Von Karman (Eckert 1959) expresses the relation that must exist between the overall rate of flux of momentum across a section of the boundary layer, the frictional stress at the surface and the pressure gradient. It can be simply expressed as:

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho u_\infty^2} \tag{5}$$

Many researchers employed different relations for the velocity profile through the boundary layer. Von Karman (Massey 2006) assumed that the velocity profile is a polynomial function of the vertical distance. Thus,



$$u = a + b(y) + c(y)^2 \tag{6}$$

Where a, b and c are constants. The boundary conditions which are used to find the values of these constants are:

$$\text{at } y=0, \quad u=0 \tag{7}$$

$$\text{at } y = \delta, \quad u = u_\infty \tag{8}$$

shear stress (τ) decreases linearly and becomes zero when $y = \delta$ $\tag{9}$

Applying these conditions yields:

$$\frac{u}{u_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \tag{10}$$

Another velocity profile derived by Von Karman (Eckert 1959), by using four conditions. These conditions are:

$$\text{at } y = 0, \quad u = 0 \tag{11}$$

$$\text{at } y = \delta, \quad u = u_\infty \tag{12}$$

$$\text{at } y = \delta, \quad du/dy = 0 \tag{13}$$

$$\text{at } y = 0, \quad d^2u/dy^2 = 0 \quad (\text{for constant pressure condition}) \tag{14}$$

The simplest function used to satisfy these conditions is a polynomial with four arbitrary constants. Thus:

$$u = a + b(y) + c(y)^2 + d(y)^3 \tag{15}$$

After applying these conditions, a velocity profile is obtained as:

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{16}$$

Noting that, the fourth condition in eqn.14 which is used in deriving eqn.16 is not used in deriving eqn.10.

(Pohlhausen 1921), suggested that the velocity profile is a polynomial of fourth order. The profile is changed due to the variation in pressure gradient along the flat plate. For zero incidence case, the velocity profile becomes:

$$\frac{u}{u_\infty} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \tag{17}$$

The most famous solution for the laminar boundary layer is obtained by (Blasius 1908), which is called the exact solution. This solution depends on obtaining new dependent and independent variables. A non-linear third order ordinary differential equation formed due to these variables then the solution achieved without the need of mathematical expression for the velocity profile. The result of this solution is agreed, and the results of other solutions are compared with it. **Table 1** shows the comparison between these solutions.

EVALUATION OF THE GENERAL VELOCITY PROFILE

The approach used in present work is a modified approach used by Von Karman (Eckert 1959). A new condition is employed in addition to the other four conditions described in eqn.11 to eqn.14. The new condition is,

$$\text{At } y = 0 \quad \frac{\tau_w}{\mu} = \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{18}$$

So the function which can be chosen to satisfy the five boundary conditions is a fifth order polynomial with five arbitrary constants. Thus,

$$u = c_1 + c_2(y) + c_3(y)^2 + c_4(y)^3 + c_5(y)^4 \tag{19}$$

Applying boundary conditions yields:

$$C_1 = 0 \tag{20}$$

$$C_2 = \frac{\tau_w}{\mu} \tag{21}$$

$$C_3 = 0 \tag{22}$$

$$C_4 = \frac{4}{\delta^3} u_\infty - \frac{3}{\delta^2} \frac{\tau_w}{\mu} \tag{23}$$

$$C_5 = -\frac{3}{\delta^4} (u_\infty - \frac{2\delta}{3} \frac{\tau_w}{\mu}) \tag{24}$$

Substitute in eqn.19, yields:

$$u = \frac{\tau_w}{\mu} (y - \frac{3}{\delta^2} y^3 + \frac{2}{\delta^3} y^4) + \tag{25}$$

$$4u_\infty (\frac{y}{\delta})^3 - 3u_\infty (\frac{y}{\delta})^4$$

For the assumed velocity profile in eqn.19, the first term is equal to zero (C1 = 0). The velocity gradient at the wall can be expressed as;

$$\frac{du}{dy}_{y=0} = A(\frac{u_\infty}{\delta}) \tag{26}$$

Where, (A) is the proportionality constant.

Substitute eqn.26 in eqn.3, to get:

$$\frac{\tau_w}{\mu} = A(\frac{u_\infty}{\delta}) \tag{27}$$

Substitute eqn.27 in eqn.25, to get:

$$\frac{u}{u_\infty} = A \left(\left(\frac{y}{\delta}\right) - 3\left(\frac{y}{\delta}\right)^3 + 2\left(\frac{y}{\delta}\right)^4 \right) +$$

$$4\left(\frac{y}{\delta}\right)^3 - 3\left(\frac{y}{\delta}\right)^4$$

Or

$$\frac{u}{u_\infty} = A\left(\frac{y}{\delta}\right) + (4 - 3A)\left(\frac{y}{\delta}\right)^3 + \tag{28}$$

$$(2A - 3)\left(\frac{y}{\delta}\right)^4$$

This equation is the general equation for the velocity profile for laminar boundary layer over a flat plate with zero incidence.

In order to find an expression for boundary layer thickness (δ) we must use the momentum integral equation (eqn.5). Thus;

$$\frac{d}{dx} \int_0^\delta \left(\frac{u}{u_\infty}\right) \left(1 - \frac{u}{u_\infty}\right) dy = \tag{29}$$

$$\frac{\mu}{\rho u_\infty^2} \left(\frac{du}{dy}\right)_{y=0}$$

Substitute eqn.27 and eqn.28 in eqn.29, integrate, to get:

$$(0.11428 + 0.06187A - 0.03015A^2) \frac{d\delta}{dx} =$$

$$A \frac{\mu}{\rho u_\infty^2 \delta} \tag{30}$$

Eqn.30 is a differential equation, which can be solved by separating variables. The solution is:

$$\frac{\delta \sqrt{Re_x}}{x} = \left(\frac{2A}{(0.11428 + .06187A - .03015A^2)} \right)^{0.5} \tag{31}$$

VALIDATION

Eqn.28 is an original and a general equation for the velocity profile. It can be reduced to many profiles obtained by other researchers.

Von Karman (Eckert 1959) suggested that the velocity profile is a polynomial of third order. Equating the coefficient of the fourth power term in eqn.28 to zero, yields:

$$2A - 3 = 0$$

Or,

$$A = 3/2 \quad (32)$$

Substituting this value in eqn.28, to get,

$$\frac{u}{u_\infty} = \frac{3}{2} \left(\frac{y}{\delta}\right) - \frac{1}{2} \left(\frac{y}{\delta}\right)^3$$

Pohlhausen (Pohlhausen 1921) suggested that the velocity profile is a polynomial of fourth order. It is clear that the coefficient of the fourth power term in eqn.17 is equal to one. Equating the coefficient of the fourth power term in eqn.17 and eqn.28, yields:

$$2A - 3 = 1$$

Or,

$$A = 2 \quad (32)$$

Substituting this value in eqn.28, to get,

$$\frac{u}{u_\infty} = 2 \left(\frac{y}{\delta}\right) - 2 \left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$$

This is exactly the velocity profile obtained by Pohlhausen (eqn.17)

EVALUATING OF BLASIUS VELOCITY PROFILE

In the exact solution, Blasius (Blasius 1908) found a numerical expression for (δ) , (δ^*) , (θ) and (C_f) , with no explicit

mathematical expression for the velocity profile. One of the consequences of Blasius solution is,

$$\frac{\delta \sqrt{Re_x}}{x} \approx 5 \quad (33)$$

The result of the present work could be used for obtaining velocity profile agrees with exact solution. By comparing eqn.31 with eqn.33, simply result obtained:

$$\left(\frac{2A}{(0.11428 + 0.06187A - 0.03015A^2)} \right)^{0.5} \approx 5 \quad (34)$$

Solving eqn.21 for (A), yields:

$$A = 1.67326 \quad (35)$$

Substituting the value of (A) in eqn.28 to get the velocity profile for the exact solution,

$$\frac{u}{u_\infty} = 1.67326 \left(\frac{y}{\delta}\right) - 1.01978 \left(\frac{y}{\delta}\right)^3 + 0.34652 \left(\frac{y}{\delta}\right)^4 \quad (36)$$

Fig.3 shows the velocity profile for the present work and other profiles. It is clear that the profile of eqn.36 is the nearest one to the exact profile.

CONCLUSIONS

The result of this work is a new velocity profile for laminar boundary layer over a flat plate with zero incidence. This profile is nearest to the exact solution obtained by Blasius (Blasius 1908). Also the consequences of using the new profile in calculating (δ) , (δ^*) , (θ) and

(C_f) are mostly nearest to the corresponding values of the exact solution as shown in **table 1**.

The new profile is mathematically simple, accurate and including most conditions bounded the laminar boundary layer. Therefore it can be used in fluid mechanics and convective heat transfer fields.

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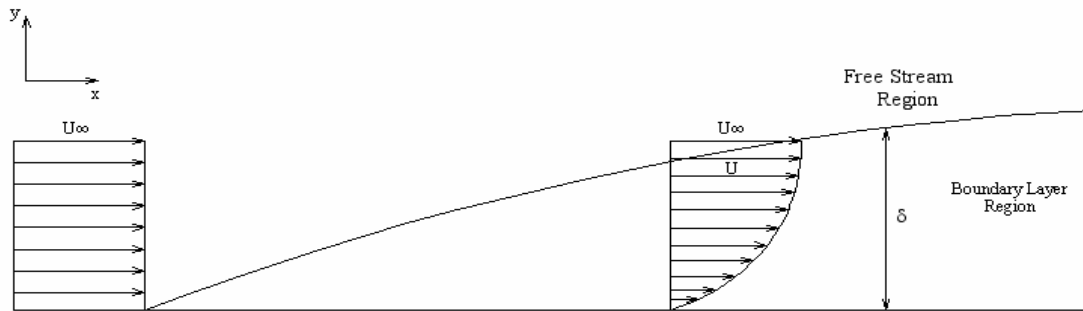


Figure 1: Velocity variation inside the boundary layer region

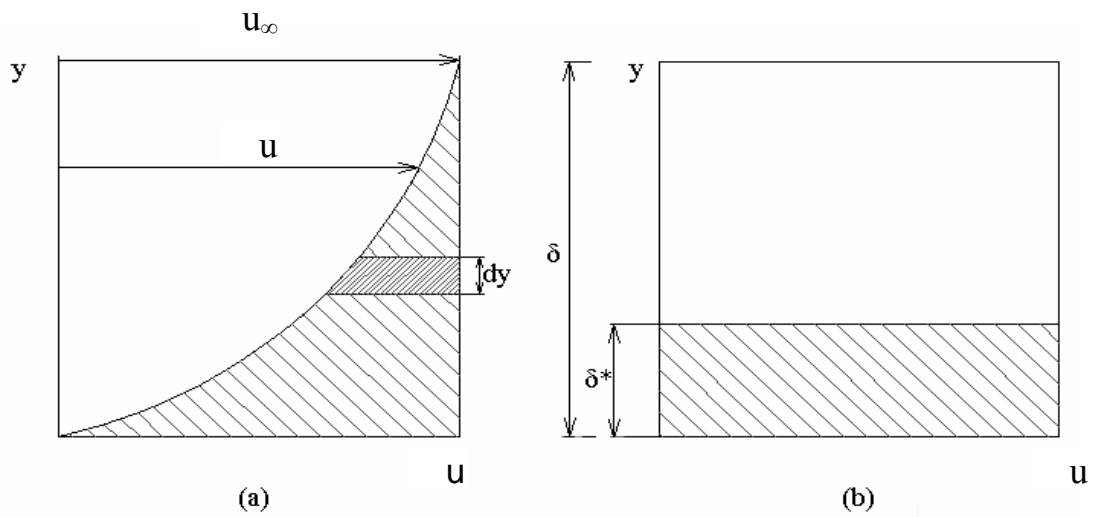


Figure 2: Observation of the displacement thickness (δ^*)

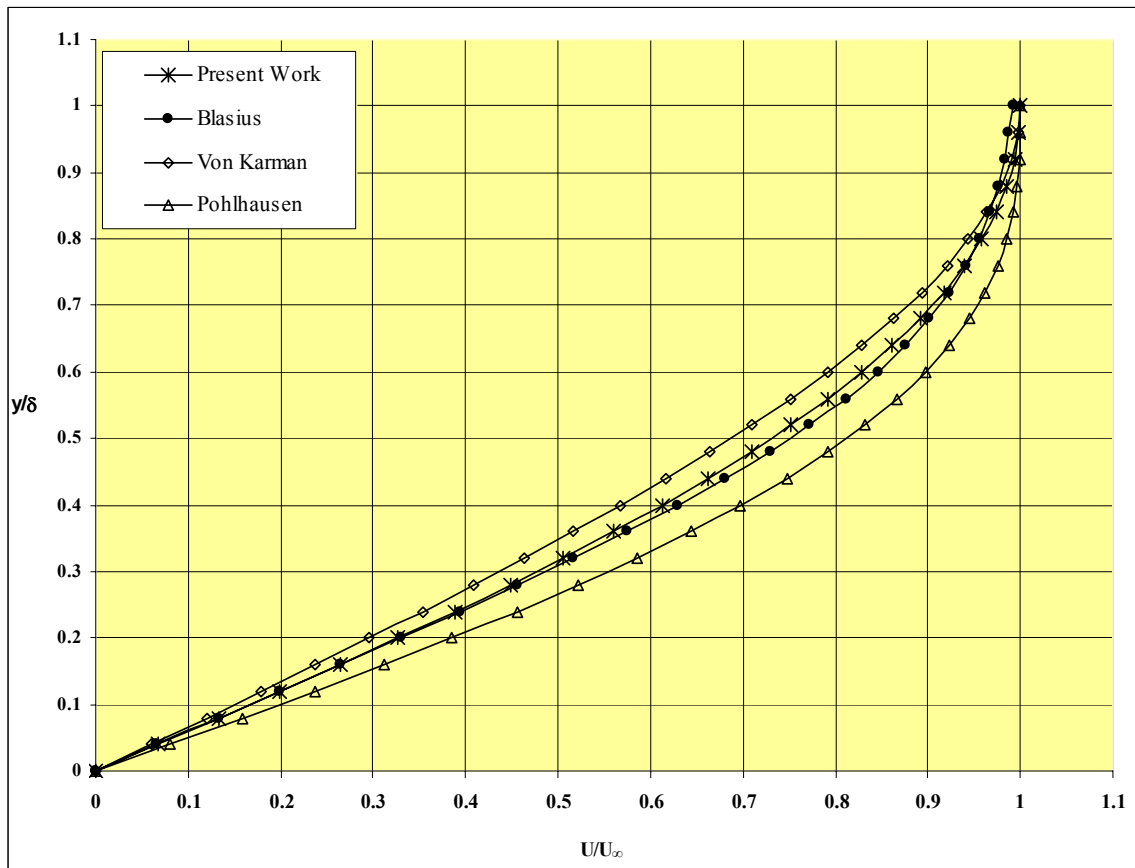


Figure 3: Velocity Profiles for the Present Work and Previous Studies Compared with Blasius Solution

Table 1: Results of δ , δ^* , θ and C_f for Number of Obtained Velocity Profiles

$\frac{u}{u_\infty}$	$\frac{\delta\sqrt{R_{ex}}}{x}$	$\frac{\delta^*\sqrt{R_{ex}}}{x}$	$\frac{\theta\sqrt{R_{ex}}}{x}$	$\frac{C_f\sqrt{R_{ex}}}{2}$
$\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	4.64	1.74	0.646	0.323
$2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$	5.84	1.752	0.686	0.343
Blasius solution	5	1.721	0.664	0.332
Present work	5	1.745	0.667	0.334

**Nomenclatures**

Symbol	Meaning	Unit
C_f	Coefficient of friction	-
Re	Reynolds number	-
u	Flow velocity	m/s
u_∞	Free stream velocity	m/s
y	Vertical distance	m
ρ	Flow density	kg/m ³
δ	Boundary layer thickness	m
δ^*	Displacement thickness	m
θ	Momentum thickness	m
μ	Dynamic viscosity	N.s/m ²
τ_w	Wall shear stress	N/m ²