FREE VIBRATION ANALYSIS OF STIFFENED CONICAL SHELL

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ABSTRACT
This paper presents a procedure for the free vibration analysis of stiffened conical shell by the finite element method. The element used is a modified eight-node superparametric shell element. The effects of the number and cross-section area of stiffeners on the conical shells were analyzed. The results showed that increasing the number of stiffeners and their cross-sectional area tend to increase the natural frequency of the conical shell. These results are compared with available research results and those obtained from MSC\NASTRAN .

الخلاصة
في هذا البحث تم وضع طريقة لتحليل الاهتزاز الحر للقشريات المخروطية المدعمة باستخدام طريقة العناصر المحددة. العنصر الذي تم استخدامه في هذه الطريقة هو عنصر قشري بارامتري فائق ذو ثماني عقد.
لقد تمت دراسة تأثير كل من عدد الدعامات المستخدمة ومساحة المقطع لها على التردد الطبيعي للقشريات المخروطية. لقد تبين من النتائج بان زيادة عدد الدعامات ومساحة المقطع لها سوف تؤدي إلى زيادة مقدار التردد الطبيعي للقشريات المخروطية.
لقد تم مقارنة هذه النتائج مع نتائج من بحوث منشورة ومع النتائج التي تم الحصول عليها من برنامج ناستران.

KEY WORDS
Conical shells, Finite element, FREE VIBRATION ANALYSIS

INTRODUCTION
Knowledge of the free vibration characteristics of elastic shells is important to the general understanding of the fundamental behavior of shells and to the industrial applications of these structures. In connection with the latter, the natural frequencies of the shell must be known in order to avoid the destructive effect of resonance with nearby rotating or oscillating equipments or other dynamic excitations such as earthquakes. [Garnet and Kempner 1964] found, by means of Rayleigh-Ritz procedure, the lowest axisymmetric modes of vibration of truncated conical shells. Transverse shear deformation and rotatory inertia effects were accounted for and the results were compared with those predicted by the classical thin-shell theory. Elemental mass matrices have been produced by [Ross 1975] for the free vibration of conical and cylindrical shells, based on a semi-analytical approach. Frequencies and modes of vibration have been compared with existing solutions and also with experimental results obtained from other sources. [Irie, Yamada and Kaneko 1982] presented an analysis for free vibration of a truncated conical shell with variable thickness by use of the transfer matrix approach. The applicability of the classical thin shell theory was assumed and the governing equations of vibration of a conical shell were written as a coupled set of first order differential equations by using the transfer matrix of the shell. The natural frequencies and the

mode shapes of vibration were calculated numerically in terms of the elements of the matrix under any combination of boundary conditions at the edge. [Ansar, Yam and lee 1985] studied the axisymmetric and asymmetric responses to free and forced vibrations of various types of shells of revolution through the finite element analysis utilizing curved and/or conical elements. A computer program package was developed and it was utilized to investigate the vibration characteristics of bells. [Mustafa and Ali 1987] presented a work in which the application of structural symmetry techniques to the free vibration analysis of cylindrical and conical shells for the prediction of natural frequency and mode shapes was described. Half and quarter models of the shell were developed and analyzed using semi-loaf and facet shell finite elements. Stiffened and unstiffened circular cylindrical and conical shells were considered. [David, Thambiratnam and Thevendran 1964] studied the optimum design of conical shells for free vibration. The lowest frequency was considered, results indicate considerable elevation in frequencies for the shells restrained at the base and free at the top. A numerical procedure incorporating the optimization technique and the finite element method was used.

The present work consists of the free vibration analysis of stiffened conical shells, taking into consideration the effect of the number, size and shape of stiffeners on the natural frequency of stiffened conical shells. This analysis was carried out via the finite element method, a special purpose computer program was built to achieve such a task. In order to support the results obtained from this computer program, a comparison was made with the results of [Mustafa and Ali 1987] and the results obtained by running MSC\NASTRAN package.

FORMULATION OF SUPERPARAMETRIC SHELL ELEMENT

This element consists of four corner and four midside nodes as shown in **Fig. (1)**. The degrees of freedom considered at the nodes are the three translations u, v, w of the midsurface and two rotations α and β of the normal to the midsurface as shown in **Fig.(2)**. The Cartesian coordinate of any point of the shell and the curvilinear coordinate can be written in the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum N_i \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}_{middle} + \sum N_i \frac{\zeta}{2} \begin{bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{bmatrix} h \qquad (1)$$

Where h the thickness of element and l_{3i}, m_{3i} and n_{3i} are the direction cosines. Here N_i is a function taking a value of unity at the node i and zero at all other nodes is called as "shape function "[William Weavr and Paul R. Johnston 1987], as shown in **Table.(1)**.

In the kinematics formulation two assumptions are imposed:

1- Nodal fiber is inextensible.

2- Only small rotations are considered.

The displacements at any point (ξ, η, ζ) can be expressed in terms of the nodal displacements as

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{8} N_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum_{i=1}^{8} N_i \zeta \frac{h_i}{2} \mu_i \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \qquad (2)$$

In this formula the symbol μ_i denotes the following matrix:

$$\mu_i = \begin{bmatrix} -l_{2i} & l_{1i} \\ -m_{2i} & m_{1i} \\ -n_{2i} & n_{1i} \end{bmatrix}$$

Column 1 in this array contains negative values of the direction cosines of the second tangential vector V_{2i}, and column 2 has the direction cosines for the first tangential vector V_{1i}.
The assumptions devoted to the used element are:

1- The strain in the direction normal to the mid-surface is assumed to be negligible ($\varepsilon_{z'}$)
2- A normal to mid-surface of the shell element will remain normal to the mid-surface of the shell after deformation.

The displacement shape functions may be cast into the matrix form:

$$[N_i]=[N_{Ai}]+\zeta\ [N_{Bi}] \quad (i=1,2\ldots8) \tag{3}$$

Where

$$[N_{Ai}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} N_i \quad \text{and} \quad [N_{Bi}]=\begin{bmatrix} 0 & 0 & 0 & -l_{2i} & l_{1i} \\ 0 & 0 & 0 & -m_{2i} & m_{1i} \\ 0 & 0 & 0 & -n_{2i} & n_{1i} \end{bmatrix} \frac{h_i}{2} N_i \tag{4}$$

The 3 X 3 Jacobian matrix required in this formulation is:

$$[J]=\begin{bmatrix} x,\xi & y,\xi & z,\xi \\ x,\eta & y,\eta & z,\eta \\ x,\zeta & y,\zeta & z,\zeta \end{bmatrix}. \tag{5}$$

The derivatives in matrix [J] can be found from eq.(1)

$$x_{,\xi} = \sum_{i=1}^{8} N_{i,\xi} x_i + \sum_{i=1}^{8} N_{i,\xi} \zeta \frac{h_i}{2} l_{3i}$$

$$x_{,\eta} = \sum_{i=1}^{8} N_{i,\eta} x_i + \sum_{i=1}^{8} N_{i,\eta} \zeta \frac{h_i}{2} l_{3i} \qquad \text{and so on}$$

$$x_{,\zeta} = \sum_{i=1}^{8} N_i \frac{h_i}{2} l_{3i}$$

For this element, six types of non-zero strains are,

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ v_{,z} + w_{,y} \\ w_{,x} + u_{,z} \end{bmatrix} \tag{6}$$

The stress-resultant vector in the local coordinate system is,

$$\{N'\}=\{N_{x'}\ N_{y'}\ N_{x'y'}\ Q_{y'}\ Q_{x'}\ M_{x'}\ M_{y'}\ M_{x'y'}\}^T \tag{7}$$

The relationship between the stress resultants and the generalized strains can be stated as follows,

$$\{N'\}=[D']\{\varepsilon'\} \tag{8}$$

Where [D′] is the rigidity matrix. A typical rigidity matrix is given by,

$$[D']=\frac{Eh}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{h^2}{12} & \frac{h^2\nu}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{h^2\nu}{12} & \frac{h^2}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h^2(1-\nu)}{24} \end{bmatrix} \tag{9}$$

where, k =shear correction factor (assumed k=1.2) [William Weavr and Paul R. Johnston 1987].

STIFFENED CONICAL SHELLS

If the spacing of the stiffeners is uniform and they lie along the natural coordinate directions, equivalent shell rigidities can be obtained by merging the stiffener rigidities with those of the shell. **Fig. (3)** shows a shell of thickness t with eccentric stiffeners in ξ and η directions at intervals s_ξ and s_η respectively and **Fig.(4)** shows the geometry of conical shell with stiffeners [D. N. Buragohain and A. S. Patil 1985]. **Fig.(5)** shows seven types of stiffeners which used in this work.
The kinematics relations between the displacements at the rib centroidal axis and those at the shell midsurface are given in eq. (10), in which all the displacements are along the coordinate system x′,y′,z′ defined at the point under consideration, with z′ along the thickness of the shell, x′ tangential to the stiffener along the ξ direction, and y′ tangential to the stiffener along η direction.

For ξ rib: $w_{rx'}=w'$, $u_{rx'}=u'+e_\xi(\partial u'/\partial z')$ (10a)

For η rib: $w_{ry'}=w'$, $v_{ry'}=v'+e_\eta(\partial v'/\partial z')$ (10b)

Here u′,v′,w′ are the displacement at the shell midsurface and e_ξ , e_η are the eccentricities of the stiffeners.
The stress resultants and strains of the ξ directional rib are,

$$\left.\begin{aligned}\{N_{x'}\}_r &= 1/h\,\{N_{rx'}\ Q_{rx}\ M_{rx}\ T_{rx'}\}^T \\ \{\varepsilon_{x'}\}_r &= 1/h\,\{\varepsilon_{rx'}\ \gamma_{rx'z'}\ \chi_{rx'}\ \partial\theta_{rx'}/\partial x'\}.\end{aligned}\right\} \tag{11}$$

in which $N_{rx'}$,$Q_{rx'}$,$M_{rx'}$ and $T_{rx'}$ are axial force, shear force, bending moment and torsional moment respectively.
The relation between stiffener strains and shell strains are given by,

$$\begin{aligned}\varepsilon_{rx'} &= \varepsilon_{x'O}+e_\xi\chi_{x'} \\ \chi_{rx'} &= \chi_{x'} \\ \gamma_{rx'z'} &= \gamma_{z'x'O} \\ \partial\theta_{rx'}/\partial x' &= 1/2\,\chi_{x'y'}\end{aligned} \tag{12}$$

Similar expressions can be written for η directional stiffeners. These relations for both sets of stiffeners may be expressed in matrix form as,

$$\{\varepsilon'\}_r=[T]\{\varepsilon'\} \tag{13}$$

The stress resultants in terms of strains can be written for both sets of stiffeners together as,

$$\{N'\}_r=[D']_r\{\varepsilon'\}_r \tag{14}$$

The various matrices in eqs. (13) and (14) are given by,

$\{\varepsilon'\}_r=\{\varepsilon_{rx'}\ \varepsilon_{ry'}\ \gamma_{rx'y'}\ \gamma_{ry'z'}\ \gamma_{rz'x'}\ \chi_{rx'}\ \chi_{ry'}\ \chi_{rx'y'}\}^T$ (15)

$\{N'\}_r=\{N_{rx'}\ N_{ry'}\ N_{rx'y'}\ Q_{ry'}\ Q_{rx'}\ M_{rx'}\ M_{ry'}\ M_{rx'y'}\}^T$ (16)

where,

$$[T]=\begin{bmatrix}1&0&0&0&0&e_\xi&0&0\\0&1&0&0&0&0&e_\eta&0\\0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&\frac{1}{2}\end{bmatrix} \qquad (17)$$

and,

$$[D]_r=\begin{bmatrix}E_\xi A\xi&0&0&0&0&0&0&0\\0&E_\eta A\eta&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&G_\eta S_\eta&0&0&0&0\\0&0&0&0&G_\xi S_\xi&0&0&0\\0&0&0&0&0&E_\xi I_\xi&0&0\\0&0&0&0&0&0&E_\eta I_\eta&0\\0&0&0&0&0&0&0&G_\xi J_\xi+G_\eta J_\eta\end{bmatrix} \qquad (18)$$

In eq. (18), E, G, A, S, I and J denote Young's modulus, shear modulus, cross sectional area, shear area, moment of inertia and torsional inertia respectively, of the ξ, or η directional stiffener as indicated by the subscript. In eqs. (16), the quantity $M_{rx'y'}$ give the sum of the torsional moments of both stiffeners.

If the stiffener rigidities are uniformly distributed over the spacing of the stiffeners to obtain equivalent rigidities over the shell midsurface, then from eq.(13) and (14), the strain energy of the stiffeners can be obtained as,

$$U_r=\frac{1}{2}\int\{\varepsilon'\}^T[T]^T[D]_r[T]\{\varepsilon'\}\ dA \qquad (19)$$

in which $[D]_r$ is obtained by dividing the rigidity terms corresponding to ξ ribs by s_ξ and those corresponding to η ribs by s_η in $[D']_r$.

The total strain energy of the stiffened shell is then given by,

$$U=\frac{1}{2}\int(\{\varepsilon'\}^T[D']\{\varepsilon'\}+\{\varepsilon'\}^T[T]^T[D]_r[T]\{\varepsilon'\})dA \qquad (20)$$

This is equivalent to the behavior of a homogeneous shell with equivalent rigidity matrix $[D_{eq}]$ given by.

$$[D_{eq}]=[D']+[T]^T[D]_r[T] \qquad (21)$$

The stiffened shell then can be analyzed as a homogeneous shell using the element described earlier.

To automate the stiffener spacing calculations, the following method can be implemented. Let n_ξ be the number of ξ directional stiffeners and n_η is the number of η directional stiffeners within the element. At a Gauss point, the following two values are computed

$$S_1 = 2(J_{21}^2 + J_{22}^2 + J_{23}^2)^{1/2}$$
$$S_2 = 2(J_{11}^2 + J_{12}^2 + J_{13}^2)^{1/2} \qquad (22)$$

in which J's are the coefficients of the Jacobian matrix. Effectively S_1 gives the dimension of the element along the η direction and S_2 gives the dimension along the ξ direction at that Gauss point. Thus,

$$S_\xi = {S_1}/{n_\xi}$$
$$S_\eta = {S_2}/{n_\eta} \qquad (23)$$

SOLUTION OF EQUATION

The equation of motion for a zero external force vector R can be presented as,

$$[M]\{\ddot{U}\}+[K]\{U\}=0 \qquad (24)$$

For harmonic displacements,

$$U_i = \Phi_i \sin(\omega_i t+\alpha_i) \qquad i=1,2,\ldots DOF \qquad (25)$$

In this harmonic expression, Φ_i is a vector of nodal amplitudes (*mode shape*) for the ith mode of vibration. The symbol ω_i represents the *angular frequency* of mode i, and α_i denotes the *phase angle*. By differentiating eq. (25) twice with respect to time t ,

$$\ddot{U}_i = -\omega_i^2 \phi_i \sin(\omega_i t + \alpha_i) \qquad (26)$$

Substitution of eq.(26) and eq.(27) into eq.(24) allows cancellation of the term $\sin(\omega_i t+\alpha_i)$, which leaves,

$$(K - \omega_i^2 M)\phi_i = 0 \qquad (27)$$

Eq. (27) has the form of the algebraic eigenvalue problem.
The most efficient form of eq. (27) for structural vibrations accepts the eigenvalue problem only in the following standard,

$$(A - \lambda_i I)XX_i = 0 \qquad (28)$$

In which (A) is a symmetric matrix (dynamic matrix) and (I) is an identity matrix. the symbol λ_i denotes the ith eigenvalue, and XX_i is the corresponding eigenvector for a new system of homogeneous equations. Eq.(27) can be written in the form of eq.(28) by factoring either matrix [K] or matrix [M], using the *Cholesky square root method,* which makes use of the fact that any square matrix [A] can be expressed as the product of an upper and lower triangular matrix.

RESULTS AND DISCUSSIONS

The vibration characteristics of a conical shell is important to understand the fundamental behavior of shells and the industrial applications of these structures. According to [Mustafa B. A., Ali R. 1987], one quarter of the model has been analyzed. The shape of stiffened (ring and string) conical

shell and material properties are shown in **Fig.(6)**. The boundary condition was zero translation at ends i.e. (shear diaphragm ends).
A convergence test was made in order to select a suitable mesh size. **Fig.(7)** shows the variation of natural frequency with total degrees of freedom. It was notice that the natural frequencies were stabilized after 100 degrees of freedom.
The results of the present work were compared with the theoretical results of [Mustafa B. A., Ali R. 1987] and with those obtained from MSC\NASTRAN as shown in **Table (2)**.
The percentage errors were computed between the present work ,MSC\NASTRAN, and [Mustafa B. A., Ali R. 1987], and recorded in **Table (2)**. It was noted that these percentage errors were small because the special shell element (i.e. superparametric shell element) was used, which have 40 degrees of freedom. The natural frequency increases when the conical shell thickness increase as shown in **Fig.(8)**. The conical shell becomes less stiff and hence the frequency decreases. It was noted from **Fig.(9)** that when the shell was stiffened the natural frequency was increased according to the number of stiffeners (string) i.e. (natural frequency increases when increasing number of stiffeners). And the same results were obtained for ring stiffeners as shown in **Fig.(10)**. In general, the stiffeners were required to increase the bending stiffness of such thin walled members (shells and plates). Consequently, stiffened shells were often used in aircraft and launch vehicles to obtain lightweight structures with high bending stiffness and decreasing in the mode shape. So, the suitable numbers of stiffeners were 3 for rings and 3 for strings. **Table (3)** shows the results of natural frequencies for each shape in **Fig.(6)** but with constant cross-sectional area. These stiffeners have a great effect on results, for which the maximum resistance stresses of each stiffener, due to bending, were proportional to the distances of the most remote fibers from the neutral axis of the cross section. In order to obtain the maximum resistance to bending, sections with large area far away from the neutral axis are implemented. It was noted that the natural frequency increases with increasing the cross- sectional area as shown in **Fig.(11)**. The increase in cross-sectional area caused an increase in the structure stiffness. From **Fig.(12)** it was recognized that the natural frequency increases with decreasing the angle of the conical shell. Thus decreasing the cone angle caused a decrease in the structure stiffness .

CONCLUSIONS:

The conclusions obtained from the present analysis can be summarized as follows:

1- The effect of the thickness on the natural frequency is studied and it is noted that the smallest natural frequency occur when the thickness of conical shells was small.
2- Stiffeners and their shapes have a great effect on natural frequencies, where the natural frequency increases with increasing the number of stiffeners and their cross-sectional area.
3- Increasing the cone angle tends to reduce the natural frequency.
4- It can be seen that the superparametric shell element gives good results in such vibrational analysis of stiffened conical shell.

Table (1) Shape Function for Midsurface Interpolation of Shell Elements

Serendipity 8-node element:
Corner nodes: $N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$
Midside nodes: $N_i = \frac{1}{2}\xi_i^2(1+\xi\xi_i)(1-\eta^2)+\frac{1}{2}\eta_i^2(1+\eta\eta_i)(1-\xi^2)$

Table (2) Verification Test for vibration case

	Natural Frequency (Hz)
Present work	1352.181
MSC\NASTRAN	1361,54
[Mustafa B. A., Ali R. 1987]	1333
Percentage Error	1.425%

Table (3) Type of different stiffeners for vibration case

Shape No.	**I*10^{-8} (m^4)**	**J*10^{-8} (m^4)**	**Natural Frequency (Hz)**	
			Present work	**MSC\ NASTRAN**
1	0.4	0.8	1138.13	1154.22
2	2.4	2.6	1188.67	1214.53
3	2	2.2	1182.44	1222.36
4	2	2.1664	1181.79	1220.13
5	3.4	6.6	1201.89	1251.91
6	2.2	13.6	1206.12	1259.34
7	2	2.2	1182.44	1222.36

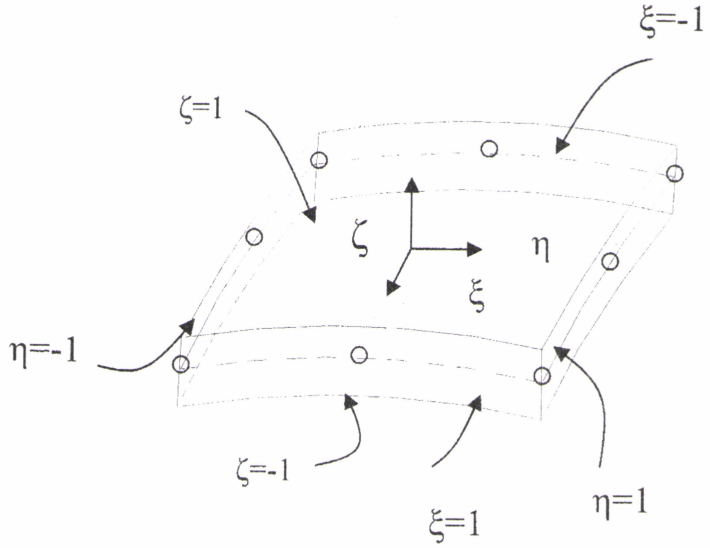


Fig. (1) Eight nodded shell element

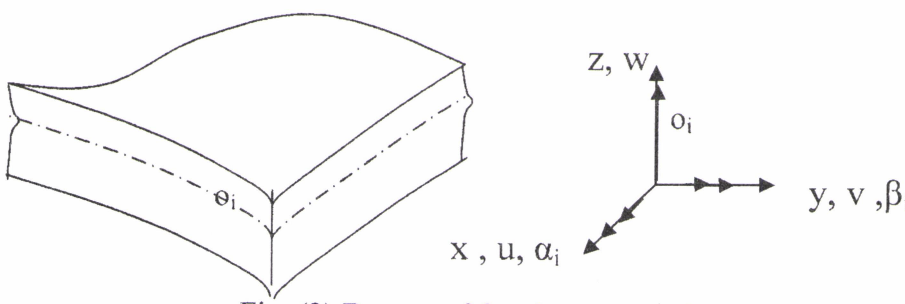
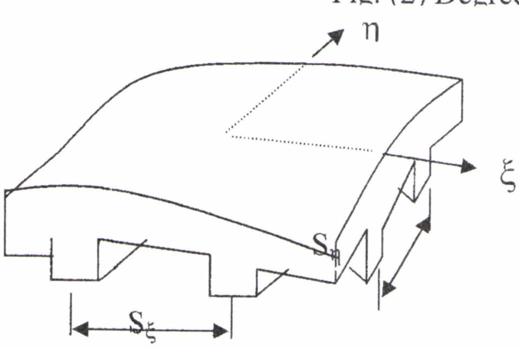


Fig. (2) Degree of freedom at a node



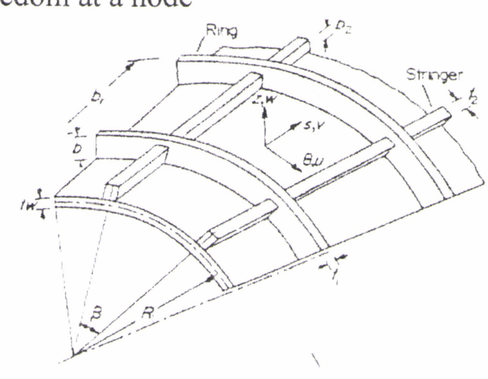


Fig. (3) Stiffened shell

Fig. (4) Geometry of conical shell with stiffeners

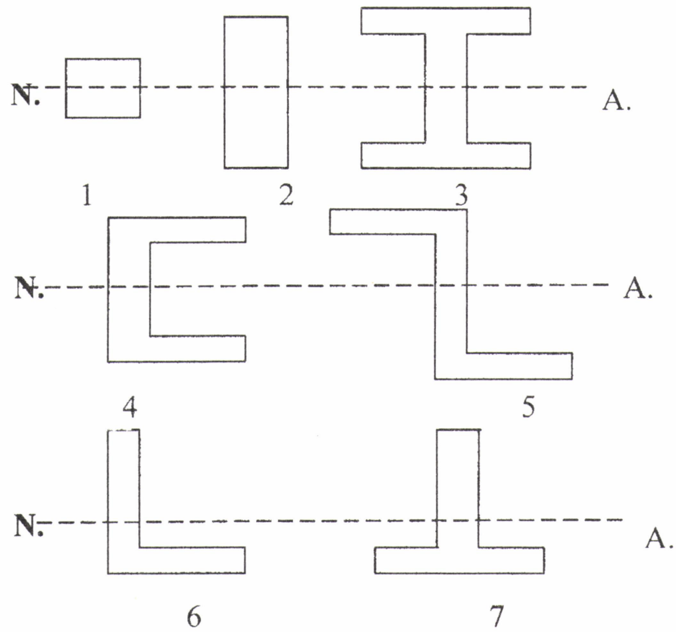
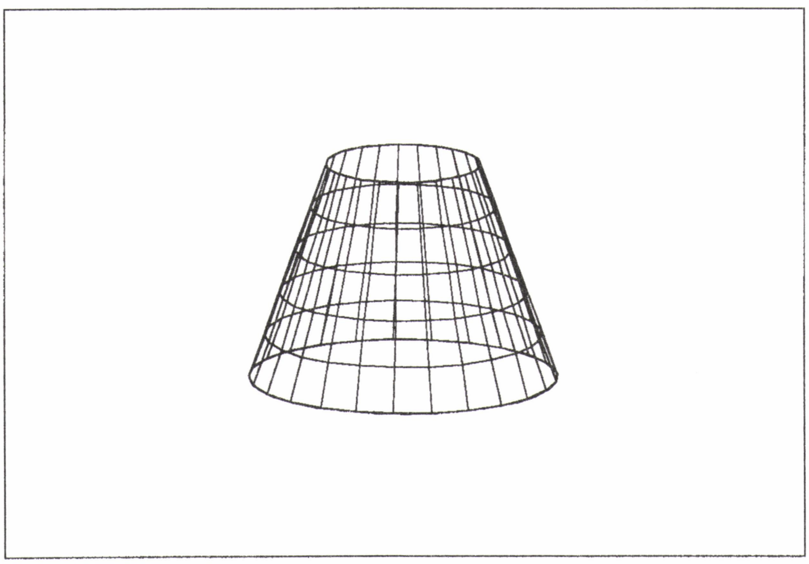


Fig.(5) Type of stiffeners



Length=0.2667 m
Small radius=0.087 m
Large radius=0.134 m
Thickness=2.54 mm
E= 68.95E9 N/m^2
ν= 0.303
ρ= 2714 Kg/m^3
σ_y=2.4E8 N/m^2

Ring depth= 6.35E-3 m
Ring width= 6.35E-3 m
E_r=68.95E9 N/m^2
ρ_r= 2714 Kg/m^3
No. of rings=3
σ_{yr}=2.4E8 N/m^2

String depth= 1.27E-2 m
String width= 6.35E-3 m
E_s=73.13E9 N/m^2
ρ_s= 2765 Kg/m^3
No. of strings=3
σ_{ys}=4.6E8 N/m^2

Fig. (6) Shape of stiffened conical shell (ring and string) with material properties.

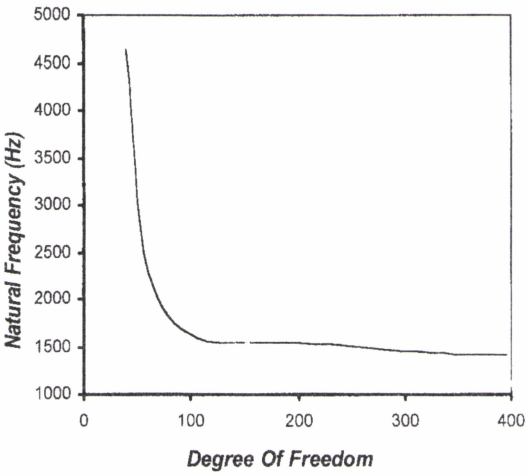


Fig.(7) Variation of Natural Frequency with total degree of freedom

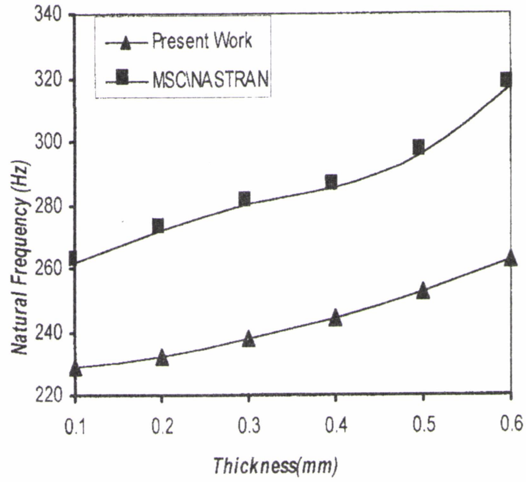


Fig.(8) Variation of natural frequency with shell thickness

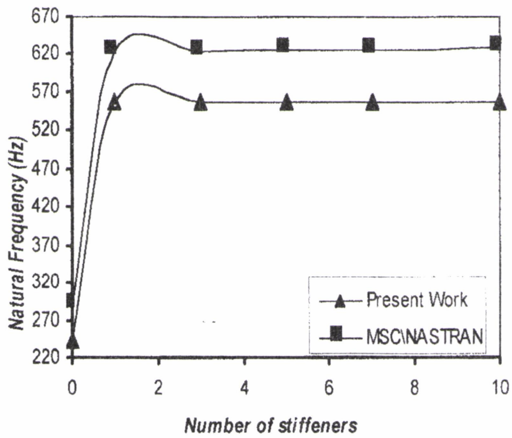
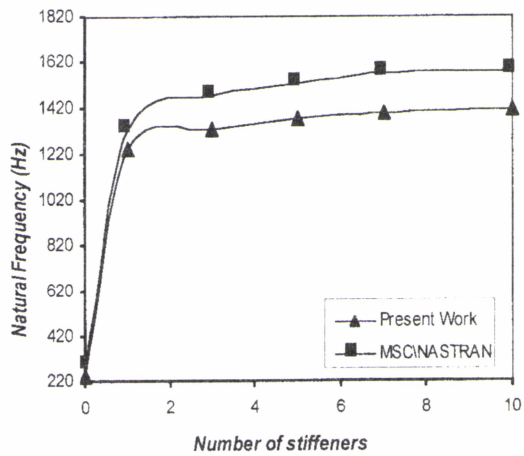
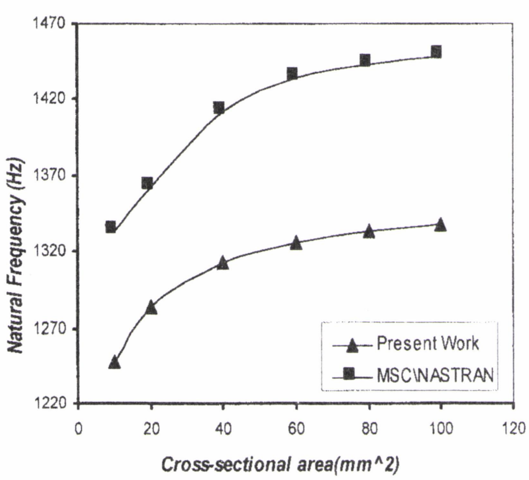


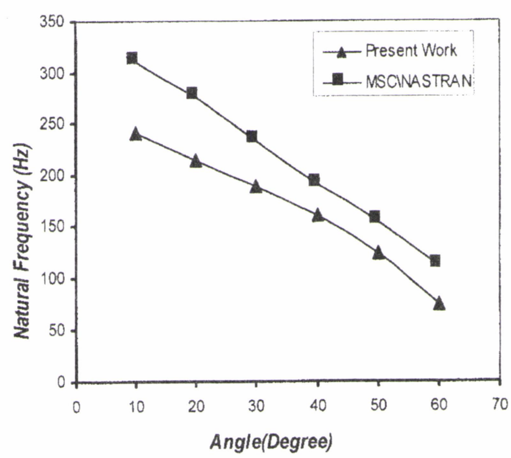
Fig. (9) Variation of natural frequency with the number of stiffeners(string)



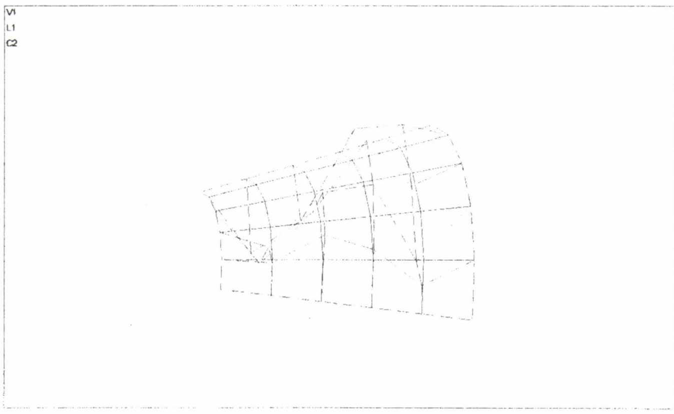
Fig(10) Variation of natural frequency with the number of stiffeners(ring)



Fig(11) Variation of natural frequency with cross-sectional area of stiffeners



Fig(12) Variation of natural frequency with the angle of cone



Fig(13) Distribution of elements inside a quarter of conical shell with it's deformed shape for the first mode

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NOTATION

E	Young's modulus	N/m^2
t	Thickness	m
ν	Poisson's ratio	—
A_r	Area of ring	m^2
A_s	Area of string	m^2

n_r	No. of rings	—
n_s	No. of stringers	—
S_s	Shear Area of the Stringer	m^2
S_r	Shear Area of the Ring	m^2
[K]	Element Stiffness matrix	—
[M]	Element Mass matrix	—
L	Length of conical shell	m
M	Bending Moment per Unit Length	N.m/m
N	Force per Unit Length	N/m
Q	Shear per Unit Length	N\m
T	Torsion Moment per Unit Length	N.m/m
I	Bending Moment of Inertia	kg . m^2
J	Torsion Moment of Inertia	kg . m^2