DESIGN OF A SLIDING MODE CONTROLLER
FOR A TORA SYSTEM

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ABSTRACT

This paper presents the design of a sliding mode controller for an uncertain model of a TORA system (Translational Oscillations with a Rotational Actuator) as a two DOF underactuated mechanical system. The switching function is selected to make the TORA system an asymptotically stable mass-spring system with a nonlinear damping effect when the sliding mode controller constrains the state to the sliding manifold. This sliding mode controller is derived here according to a new formula consisting of continuous and discontinuous parts. The main obstacle in the controller design is the uncertainty in the switching function, which appears in the controller formula. The sliding controller is found effective in bringing the system state to the neighborhood of the equilibrium point in spite of the uncertainty in the switching function. In addition, the chattering problem is solved via the use of approximate signum function.

الخلاصة:

يتناول هذا البحث تصميم مسيطر ذو شكل منزلق لمنظومة أل TORA كمنظومة تحت الدفع ذات درجتين للحرية. تم اختيار دالة التبديل لتجعل منظومة أل TORA كمنظومة كتلة-نابض مستقرة مع مخمد لا خطي وذلك عندما يحدد المسيطر الحالة إلى سطح الانزلاق. تم اشتقاق صيغة المسيطر المنزلق في هذا البحث تبعا لصيغة جديدة تتكون من جزء مستمر و جزء غير مستمر. إن العقبة الأكبر في تصميم المسيطر هي الغموض في دالة التبديل والتي تظهر في صيغة المسيطر. وجد المسيطر المنزلق فعالاً في جلب حالة المنظومة إلى منطقة تجاور نقطة الاستقرار على الرغم من الغموض في دالة التبديل. بالإضافة إلى ذالك تم حل مشكلة الارتجاج باستخدام دالة تقريبية لدالة الإشارة.

KEYWORDS: under actuated mechanical system, sliding mode controller, uncertainty

INTRODUCTION

Underactuated system is a mechanical system with number of actuators less than the number of configuration variables that describe the mechanical system behavior according to Euler-Lagrange equation. The TORA system is a two DOF mechanical system where the mass-spring system is actuated via the rotation of an eccentric mass, as shown in Fig. 1. During the past decade many nonlinear controller design techniques were applied to the TORA system. Jankovic et. al. (Jankovic et. al. 1996) applied a passivity-based approach to the TORA system. Olfati-Saber

(Olfati-Saber 2001) transformed the underactuated system to a normal form that depends on the relative degree, and then applied known control strategies like the backstepping. The same approach of Olfati-Saber was applied by Qaiser et. al. (Qaiser et. al. 2007) to the TORA system. Zhong-Ping J. and Kanellakopoulos I. (Zhong-Ping and Kanellakopoulos 2000) proposed an observer/controller backstepping design in which one of the unmeasured state appears quadratically in the state equation. The backstepping approach applied to a nonlinear system assume exact system model. A complicated formula may result due to the application of the backstepping approach as reported in reference (Jankovic et. al. 1996).
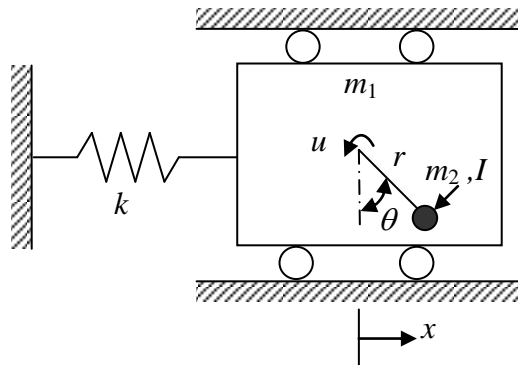

Fig 1: The TORA system.

The systems considered thus far are assumed certain systems. However, uncertainty is a major problem that has not been considered in controller design of underactuated systems thus far. Application of uncertainty is a unique, challenging and interesting research area (Spong 1997). In this work, the sliding mode controller design is accomplished for an uncertain TORA system model after transforming it to a form known as the *regular form* (Luk'yanov and Utkin 1981). A suitable switching function for the sliding controller is selected to constrain the system state to its zero-level (sliding manifold). A successful design implies that the dynamic of the underactuated system at this manifold is asymptotically stable.

Uncertainty in system parameters of an underactuated system leads to many problems like the non-linearizability, difficulty in selecting an attractive surface and the uncertainty in switching function. An attractive switching surface is proposed in reference (Hameed 2007) based on the flatness property of the TORA system. In addition, using uncertain switching function in controller formula ensures only the state is constrained in a region about the equilibrium point (Hameed 2007).

The organization of this paper is as follows: the mathematical model is derived in section two. Section three is devoted to the design of a sliding mode controller which includes the selection of switching manifold based on flatness theory, the effects of using approximate signum function and the presence of uncertainty in switching function. The new sliding mode controller formula is presented in section four and the derivation are given in appendix (A). Section five presents the results and discussion of four numerical simulation tests and finally the conclusion is presented in section six.

MATHEMATICAL FORMULATION

As shown in Fig. 1, the TORA system consists of translational oscillating platform, which is controlled via a rotational eccentric mass. The inertia matrix, potential energy and the force vector (one-form) assume the following form (Olfati-Saber 2001);

$$M = \begin{bmatrix} m_1 + m_2 & m_2 r \cos(\theta) \\ m_2 r \cos(\theta) & m_2 r^2 + I \end{bmatrix} , \ V(x,\theta) = \frac{1}{2} K x^2 + (r - r\cos(\theta)) m_2 g_0 , \ F = d\theta$$

The dynamical equation of the TORA system are (Hameed 2007):

$$\ddot{x} = \frac{1}{\Delta}\left[\left(m_2 r^2 + I_2\right)m_2 r \sin(\theta)\left(\dot{\theta}\right)^2 - \left(m_2 r^2 + I_2\right)Kx + \left(m_2 r\right)^2 g_0 \sin(\theta)\cos(\theta)\right]$$
$$\left. \quad - \frac{\left(m_2 r \cos(\theta)\right)}{\Delta}u \right\}$$

$$\left. \ddot{\theta} = \frac{1}{\Delta}\left[-\left(m_2 r\right)^2 \sin(\theta)\cos(\theta)\left(\dot{\theta}\right)^2 + \left(m_2 r\right)Kx\cos(\theta) - \left(m_1 + m_2\right)m_2 g_0 r \sin(\theta)\right] \right\} \quad (1)$$
$$\left. \quad + \frac{\left(m_1 + m_2\right)}{\Delta}u \right\}$$

where,

$$\Delta = \left(m_1 + m_2\right)\left(m_2 r^2 + I_2\right) - \left(m_2 r\right)^2 \cos^2(\theta) = m_1\left(m_2 r^2 + I_2\right) + m_2 I_2 + \left(m_2 r\right)^2 \left(\sin(\theta)\right)^2 > 0$$

Corollary (3.1) in reference (Hameed 2007) proves that the TORA system is a flat system since m_{11} is constant for the actuated shape variable. The flat output is given by;

$$y = x + \left(\frac{m_2 r}{m_1 + m_2}\right)\sin(\theta) \qquad (2)$$

Differentiating eq.(2) twice and with the aid of eq.(1) yields the underdetermined differential equation, i.e.;

$$\ddot{y} = -\left(\frac{K}{m_1 + m_2}\right)x \qquad (3)$$

Eliminating x from eqs. (2) and (3), the underdetermined equation assumes the following form;

$$\ddot{y} = -\left(\frac{K}{m_1 + m_2}\right)y + \frac{Km_2 r}{\left(m_1 + m_2\right)^2}\sin(\theta) \qquad (4)$$

The design of the sliding controller in the next item assumes the following statements;

i. The output y and its derivatives are assumed to be known precisely, which is an essential assumption.

ii. The rest points for the TORA system in terms of the configuration variables (x, θ) is the rest point set $\{(0, \theta): \theta \in (-\pi, +\pi)\}$.

According to statement (ii), the controller must be able to translate the state from any condition (any perturbed condition) to the rest point set described here only.

SLIDING MODE CONTROLLER DESIGN

The work reported in this section is based on the results of three propositions; namely propositions (4.2), (5.7) and (5.5) and lemma (5.1) of reference (Hameed 2007). The interested reader may refer to reference (Hameed 2007) for the proof of these propositions and the lemma. For convenience, however, only the proof of proposition (5.7) is presented here due to its importance.

Selection of Switching Manifold

Selection of the switching manifold is the subject of proposition (4.2), (Hameed 2007). It states that; *The selection of the function s ,where*

$$e = f_0(y, q_0) + k_1 \dot{y} + k_2 y$$
$$s = \dot{e} + \lambda e$$

will globally asymptotically stabilizes the 2DOF flat underactuated mechanical system with sliding mode controller provided that: $(i) k_1, k_2 > 0$, *and* $(ii) f_0(0, q_0) = 0 \Rightarrow q_o = 0$.

A TORA system with this switching manifold becomes globally asymptotically stable (GAS) according to proposition (5.7), (Hameed 2007). The statement of this proposition is;

Proposition: *Consider the TORA system dynamic given by eq. (1), the sliding controller* u *that use the following switching function:*

$$s = \dot{e} + e$$

$$e = \theta + \tan^{-1} c\dot{y} \tag{5}$$

Render the TORA system GAS system, where c *is a design parameter in the error function, which has a damping effect.*

Proof: To prove the validity of the switching function in eq. (5), first it is needed to show that $\ddot{y}$ is asymptotically stable (AS) with $e = 0$, i.e., if we write $\ddot{y} = f(y, \dot{y}, e)$ at any e, then $\ddot{y} = f(y, \dot{y}, 0)$ is AS. Then it is required to show that the upper sub system given by eq. (4) is (ISS) with e regarded as a disturbance. Finally it is required to show that the selection of s will not lead to a singularity in input channel i.e. $L_g s \neq 0$ at any point of (x, θ). The underdetermined equation $\ddot{y}$ given by eq. (4) may be written in state space form as;

$$\dot{y}_1 = \dot{y} = y_2$$

$$\dot{y}_2 = -\left(\frac{K}{m_1 + m_2}\right) y_1 + \frac{K m_2 r}{(m_1 + m_2)^2} \sin(\theta) \tag{6}$$

When $e = 0$, we have $\theta = -\tan^{-1}(c\dot{y})$ or $\tan(\theta) = -c\dot{y} \Rightarrow \sin(\theta) = \dfrac{-c\dot{y}}{\sqrt{1 + (c\dot{y})^2}}$, then eq. (6) becomes;

$$\dot{y}_2 = -\frac{K}{(m_1 + m_2)} y_1 - \frac{K m_2 r c}{(m_1 + m_2)^2} \frac{y_2}{\sqrt{1 + (c y_2)^2}} \tag{7}$$

Now let the Lyapunov function be given by;

$$V = \frac{A}{2} y_1^2 + \frac{1}{2} y_2^2, \ A > 0 \tag{8}$$

$$\dot{V} = A y_1 y_2 + y_2 \dot{y}_2 = A y_1 y_2 + y_2 \left[-\frac{K}{(m_1 + m_2)} y_1 - \frac{K m_2 r c}{(m_1 + m_2)^2} \frac{y_2}{\sqrt{1 + (c y_2)^2}} \right]$$

$$= \left(A - \frac{K}{m_1 + m_2} \right) y_1 y_2 - \left[\frac{K m_2 r c}{(m_1 + m_2)^2} \frac{1}{\sqrt{1 + (c y_2)^2}} \right] y_2^2 \tag{9}$$

Therefore, $\dot{V}$ is negative semi definite when A is chosen as $A = \dfrac{K}{m_1 + m_2}$ and V is a proper Lyapunov function provided that the only equilibrium point is at the origin. To show that the TORA system is ISS, re-write eq. (5) with $e \neq 0$, i.e.;

$$\dot{y}_2 = -\frac{K y_1}{(m_1 + m_2)} - \frac{K m_2 r}{(m_1 + m_2)^2} \sin\left(e - \tan^{-1}(c y_2)\right) \tag{10}$$

Clearly $\dot{y}_2$ is globally Lipstize function. It means that $g(e, y_1, y_2)$ is a bounded quantity defined as;

$$g(e, y_1, y_2) = \int_0^1 \frac{\partial \ddot{y}_2(y_1, y_2, \kappa e)}{\partial e} d\kappa \ , \ |g(e, y_1, y_2)| < L_g \ , \ L_g > 0 \tag{11}$$

The TORA system is ISS because the following integral $\int_0^\infty e \, d\tau$ is bounded since $\max_{t \in (0,\infty)} |e| < \infty$ for

$t \in (0, t_s)$. It is GAS according to theorem (4.4) in reference (Hameed 2007). Finally, $L_g s$ is greater than zero as will be shown later on when the controller is designed.

■

Approximate Signum Function

To avoid chattering, the approximate form of the signum function will be used. It is given by;

$$\text{sgn}(s) \approx \text{sgn}(s)_{app} = \begin{cases} \sin\left(\dfrac{\pi s}{2\delta_0}\right) & |s| \le \delta_0 \\ \text{sgn}(s) & otherwise \end{cases} \tag{12}$$

The effect of the *approximate signum function* on the attractiveness of the sliding manifold is governed by lemma (5.1), (Hameed 2007). It states that; *The use of the approximate signum function in eq. (12) guarantee the existence of the following attractive boundary layer around the switching manifold $s = 0$:*

$$-\delta_0 < s < \delta_0 \tag{13}$$

Using $\text{sgn}(s)_{app}$ guarantees only the attractiveness of a region bounded by a boundary layer in eq. (13).

Uncertainty in Switching Function *s*

The effect of using uncertain switching function in the sliding mode controller design will now be considered. The uncertainty in switching function is due to many reasons like the measurement error and/or the uncertainty in the output function calculation and its derivatives. Since the switching function is constructed from these quantities, then this function becomes uncertain. The uncertainty in switching function *s* is treated by proposition (5.5), (Hameed 2007). It states that: *Consider the underdetermined equation for a system given in the following form*

$$\dot{x} = f(x) + g(x)u \tag{14}$$

then the use of uncertain switching function in the controller design renders the system stable and stay in a bounded region around the origin if and only if the underdetermined equation is ISS.

SLIDING CONTROLLER FORMULA

An approach is developed in reference (Hameed 2007) for formulating sliding mode controller law for the dynamic system given by eq.(14) with scalar input *u*. The approach utilizes bounded estimation for the uncertainty in the Lie derivative of the switching function s with respect to *f* and *g* respectively. Starting from the time rate of change of the switching function s with respect to the system dynamic of eq. (14) as;

$$\dot{s} = L_f s + (L_g s) u \tag{15}$$

where $L_f s$ and $L_g s$ are the lie derivatives of s with respect to f and g, respectively. Re-writing $L_f s$ terms of its nominal value as follows;

$$L_f s = (L_f s)_{nom} + \left[L_f s - (L_f s)_{nom} \right] = (L_f s)_{nom} + \Delta(L_f s) \tag{16}$$

where, $L_f s - (L_f s)_{nom} = \Delta(L_f s)$, and;

$$\Rightarrow \left| L_f s - (L_f s)_{nom} \right| = \left| \Delta(L_f s) \right| < \delta_f \left| (L_f s)_{nom} \right| \tag{17}$$

Similarly, for $L_g s$ and noting that $L_g s > 0$;

$$L_g s = (L_g s)_{nom} + \Delta(L_g s) \tag{18}$$

$$\left| L_g s - (L_g s)_{nom} \right| = \left| \Delta(L_g s) \right| < \delta_g (L_g s)_{nom} \tag{19}$$

where $(L_f s)_{nom}$ and $(L_g s)_{nom}$ are the $L_f s$ and $L_g s$ at the system nominal parameter values and δ_f and δ_g are the percent estimation of the variation of $L_f s$ and $L_g s$ from their nominal values, respectively. Now the design of the controller law can be implemented using proposition (5.1), (Hameed 2007). The proposition states that: *Consider the uncertain dynamical model described by eq. (14), the sliding mode controller that constrains the state to the switching manifold takes the following form;*

$$u = -\left(\frac{L_f s}{L_g s} \right)_{nom} - v_0 \, \text{sgn} \, s \tag{20}$$

with v_0 equal to:

$$v_0 = 1.1 \left(\frac{\delta_f + \delta_g}{1 - \delta_g} \right) \left| \left(\frac{L_f s}{L_g s} \right)_{nom} \right| \tag{21}$$

The estimation of the upper and lower variation of $L_f s$ and $L_g s$ is not an easy task, it requires that each parameter variation should be considered separately. To overcome the problem of the uncertainty in system model, the Lie derivative of the switching function $L_f s$ is decomposed into uncertain and certain terms. Also, the uncertain term is decomposed to l terms as:

$$L_f s = \sum_{i=1}^{l} (L_f s)_{unc,i} + (L_f s)_{cer} \tag{22}$$

This approach is suitable for a more complicated type of system, which frequently appear in the underactuated mechanical system. Let each of the uncertain terms satisfy the following bounded formula:

$$\left| (L_f s)_{unc,i} - \left((L_f s)_{unc,i} \right)_{nom} \right| = \left| \Delta_i (L_f s) \right| < \delta_i \left| \left((L_f s)_{unc,i} \right)_{nom} \right|, i = 1,....,l \tag{23}$$

The difference $L_f s - (L_f s)_{nom}$ can be related to the uncertainty in eq. (23) as follows;

$$L_f s - (L_f s)_{nom} = \Delta(L_f s) = \sum_{i=1}^{l} \Delta_i (L_f s) = \sum_{i=1}^{l} \left[(L_f s)_{unc,i} - \left((L_f s)_{unc,i} \right)_{nom} \right] \tag{24}$$

Now take the absolute value of both sides:

$$\left| L_f s - (L_f s)_{nom} \right| = \left| \Delta(L_f s) \right| = \left| \sum_{i=1}^{l} \Delta_i (L_f s) \right| = \left| \sum_{i=1}^{l} \left[(L_f s)_{unc,i} - \left((L_f s)_{unc,i} \right)_{nom} \right] \right|$$

$$< \sum_{i=1}^{l} \left| (L_f s)_{unc,i} - \left((L_f s)_{unc,i} \right)_{nom} \right| < \sum_{i=1}^{l} \delta_i \left| \left((L_f s)_{unc,i} \right)_{nom} \right| = \delta_f \sum_{i=1}^{l} \left(\frac{\delta_i}{\delta_f} \right) \left| \left((L_f s)_{unc,i} \right)_{nom} \right| \tag{25}$$

where $\delta_f = \min[\delta_1,........,\delta_l]$. Add the absolute value of the certain term to the right hand side of the inequality above we get;

$$\left| L_f s - (L_f s)_{nom} \right| < \delta_f \left[\sum_{i=1}^{l} \left(\frac{\delta_i}{\delta_f} \right) \left| (L_f s)_{nom,i} \right| + \left| (L_f s)_{cer} \right| \right] = \delta_f \left| \overline{(L_f s)}_{nom} \right| \tag{26}$$

Hence, the following estimation is obtained;

$$\frac{\left| L_f s - (L_f s)_{nom} \right|}{\left| \overline{(L_f s)}_{nom} \right|} = \frac{\left| \Delta(L_f s) \right|}{\left| \overline{(L_f s)}_{nom} \right|} < \delta_f \tag{27}$$

Note that $\left| \overline{(L_f s)}_{nom} \right| > \left| (L_f s)_{nom} \right|$ and because of this inequality requirement the certain part of $L_f s$ is added.

The following general formula for the sliding mode controller design in the presence of uncertainty in system parameters can now be stated. This is the subject matter of proposition (5.2), (Hameed 2007); it states that**:** *Consider the dynamical system given in eq. (14). Assume that the Lie derivative of the switching function $s = s(x)$ with respect to f given by the form in eq. (22) and each of its ith component satisfy inequality (23), then the sliding controller that render the dynamical system AS in the presence of uncertainty in system model is given by:*

$$u = -\left(\frac{L_f s}{L_g s} \right)_{nom} - v_0 \operatorname{sgn} s \tag{28}$$

$$v_0 = 1.1 \left(\frac{\delta_f + \delta_g}{1 - \delta_g} \right) \left| \left(\frac{\overline{L_f s}}{L_g s} \right)_{nom} \right| \tag{29}$$

Appendix (A) presents sliding mode controller design details for the TORA system and the control law is given by eq. (A.22).

SIMULATION RESULTS

The results of a set of numerical experiments using the designed sliding mode controller were carried out and Figs. 2 to 15 summarize the results. Throughout the simulations, the parameters value are taken as: $m_1 = 10.25$, $m_2 = 0.97$, $r = 1.02$, $I = 1$, $K = 5.17$, and the initial conditions are; $(x, \dot{x}, \theta, \dot{\theta}) = (1,0,0,0)$. The units of the physical quantities are taken as meter, kilogram and Newton for length quantity, mass and force, respectively.

Responses shown in Figs. 2 to 7 are carried out with $c = 1$(eqs. (5) and (7)), exact signum function and without uncertainty in switching function. Figure 2 shows the position x with time where a damping like behavior is obtained with $c = 1$. This is due to the selection of the error function where the $\sin(\theta)$ term (after reaching $e = 0$) provide the damping effect. This effect stabilizes the mass-spring system. The system is highly undamped and the settling time is of the order of 200 seconds. The plots of $\dot{x}$, θ, and $\dot{\theta}$ with time are shown in Figs. 3, 4, and 5, respectively.
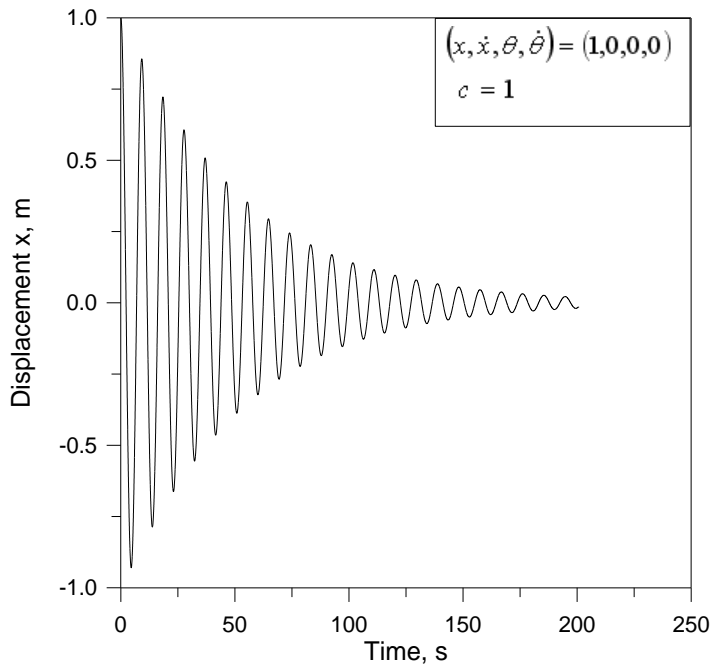
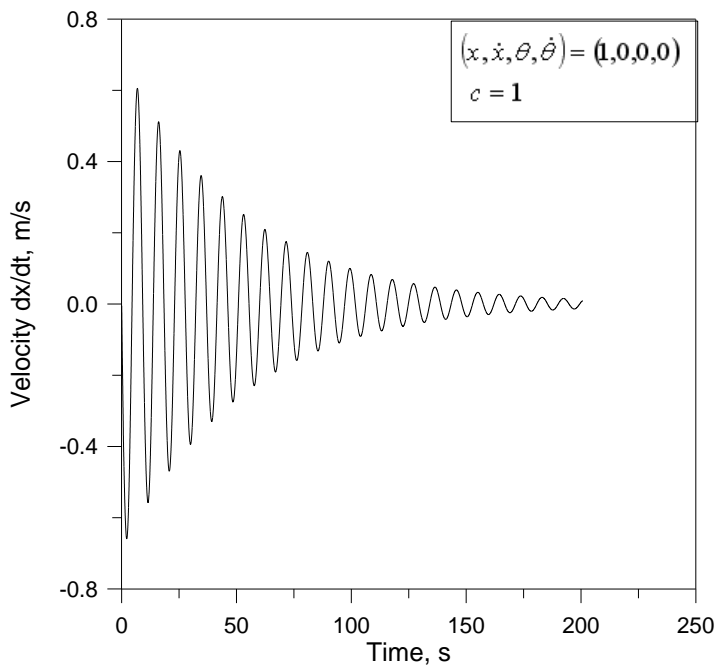**Fig. 2: Displacement versus time.**
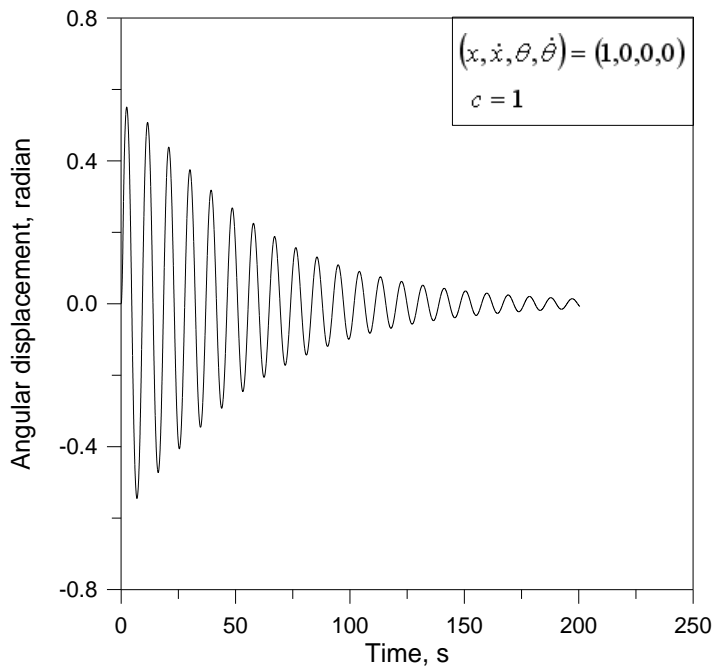

Fig. 3: Velocity versus time.

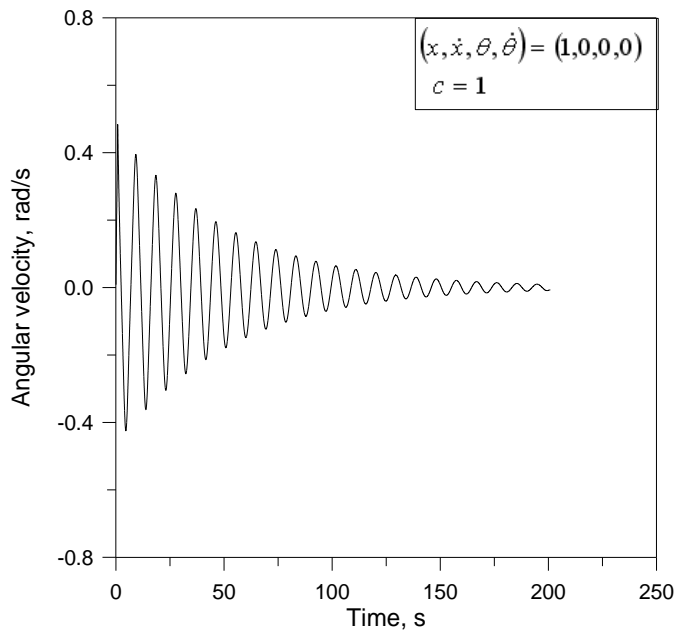**Fig. 4: Angle of eccentric mass versus time.**

Fig. 5: Angular velocity versus time.

Figure 6 shows the switching function response with time. The figure shows that the time required to reach zero-level is less than one seconds. In order to reduce the reaching time to the switching manifold a higher discontinuous gain value is required. However, this causes a high chattering effect. Also, the state is constraint to the switching manifold with very small amplitude of oscillation around it and this is because of the very small switching time interval used in the simulation. Indeed a larger value of the switching time interval causes an oscillation with higher amplitude around the switching manifold and consequently the motion is known as "zig-zag" motion (Drakunov and Utkin 1989).

The sliding control action is shown in Fig. 7. From this figure, it can be noticed that the discontinuous action nature is clear through the black region. In addition, the amplitude of the controller reduces to zero with time and it is guaranteed due to the use of variable amplitude $\left(L_f s / L_g s\right)$ (eq. (27)).
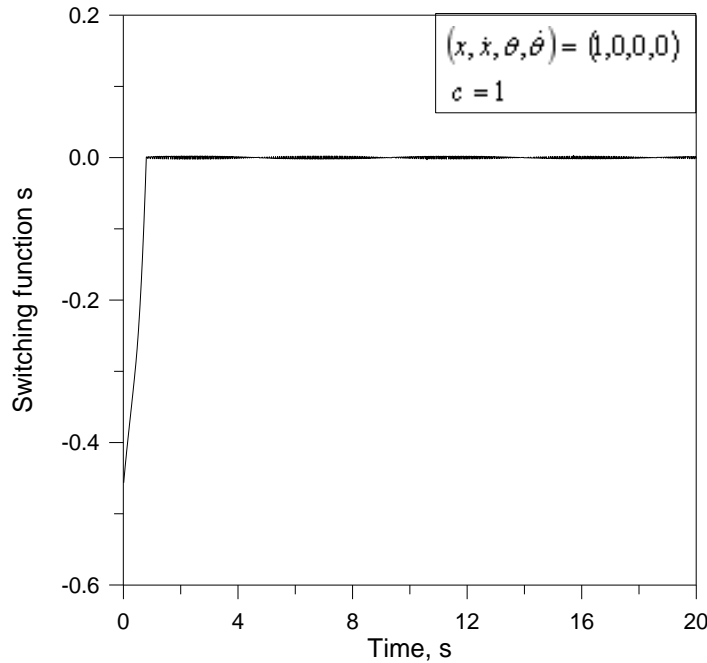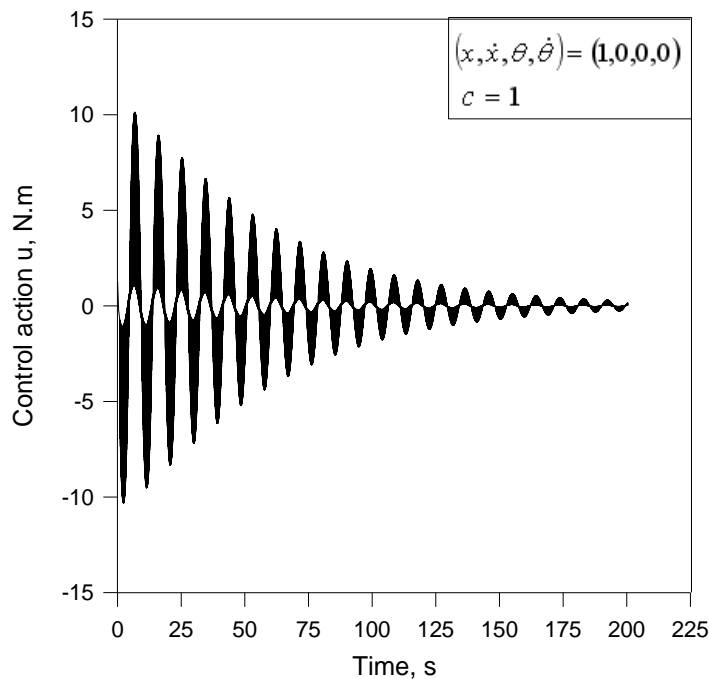

Fig. 6: Switching function versus time.

Fig. 7: Control action (torque) versus time.

For a certain model of the TORA system, Olfati-Saber (Olfati-Saber 2001) designed a nonlinear controller based on backstepping technique with certain parameter values. By utilizing these nominal values in this work, it is deduced from Fig. 8 that the displacement x is similar to those of Olfati-Saber. However, in reference (Olfati-Saber 2001) the maximum torque required do

not exceed $3\,N.m$, while in Fig. 7 the control action reaches $10\,N.m$. The increase in torque is mainly due to the uncertainty taken into account in the controller design.

Increasing the damping effect by selecting $c = 5$ will greatly improve the response, as can be clearly seen in Figs. 8 and 9. Figure 8 shows the plot of displacement x with time where the settling time is reduced to about 60 seconds. While Fig. 9 plots the control u with time. The maximum controller action increases due to higher damping property required. A comparison between Figs. 7 and 9 clearly shows the torque has increased by about 100%.
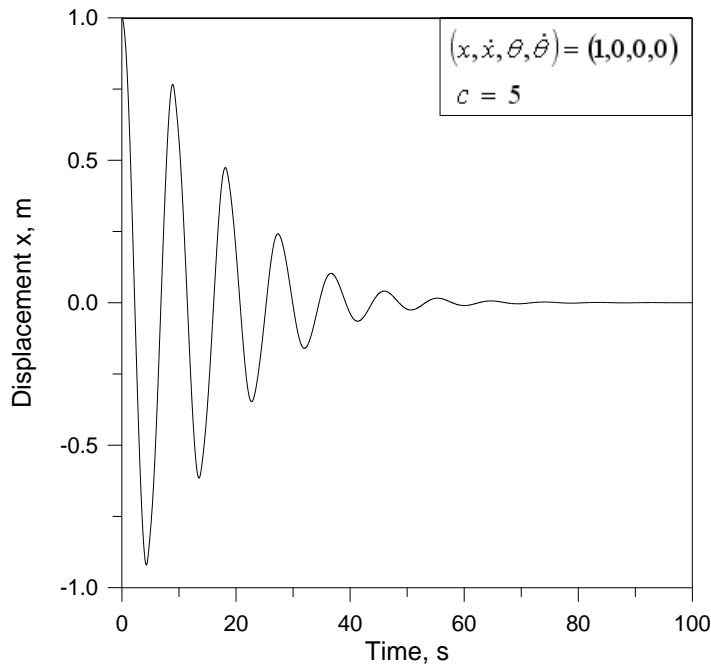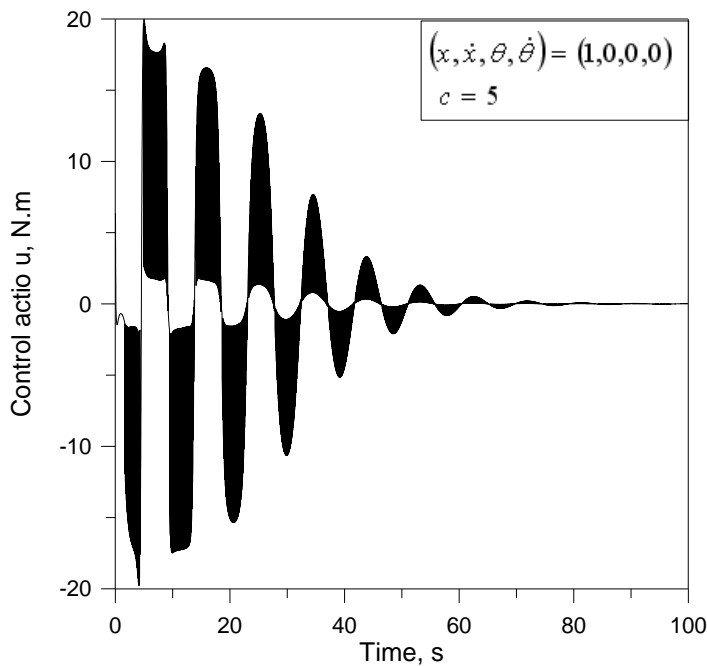

Fig. 8: Displacement versus time.

Fig. 9: Control action versus time.

A series of numerical tests were carried out where instead of the signum function the approximate one is used with $c = 5$. The plot of displacement x with time (Fig. 10) is unaffected due to the use of approximate signum function in comparison with the exact signum function shown in Fig. 8. Figures 11 and 12 show the switching function plot. A smooth sliding mode controller action versus time is obtained due to the use of the approximate switching function.

It can be observed from Fig. 11, the state stays at the switching manifold without chattering, that is smooth system behavior. A comparison between Figs. 9 and 12 reveals that the maximum torque has been reduced to approximately 10 *N.m*. This is because of the discontinuous part amplitude near the switching manifold is smaller due to the use of the approximate signum function.
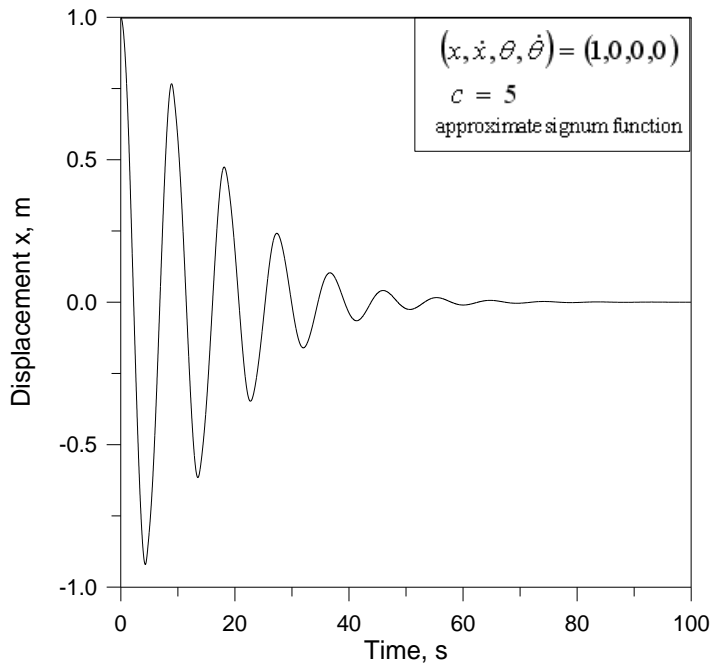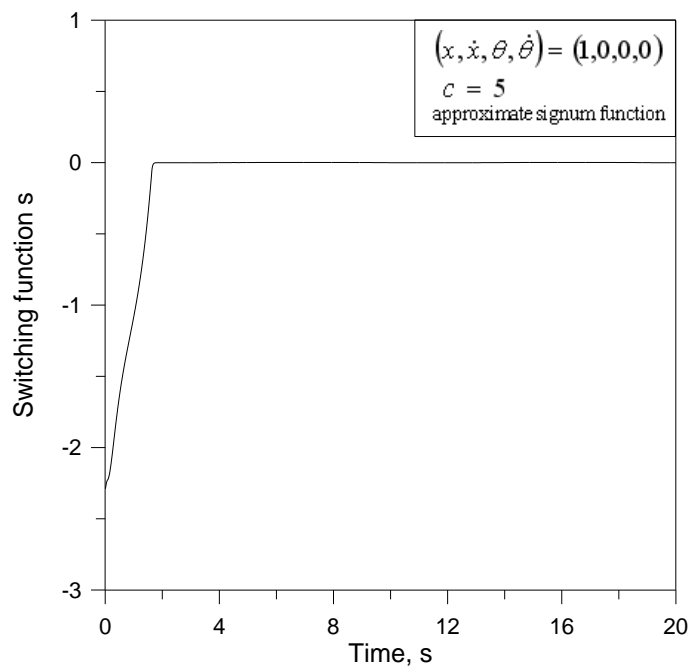

Fig. 10: Displacement versus time.

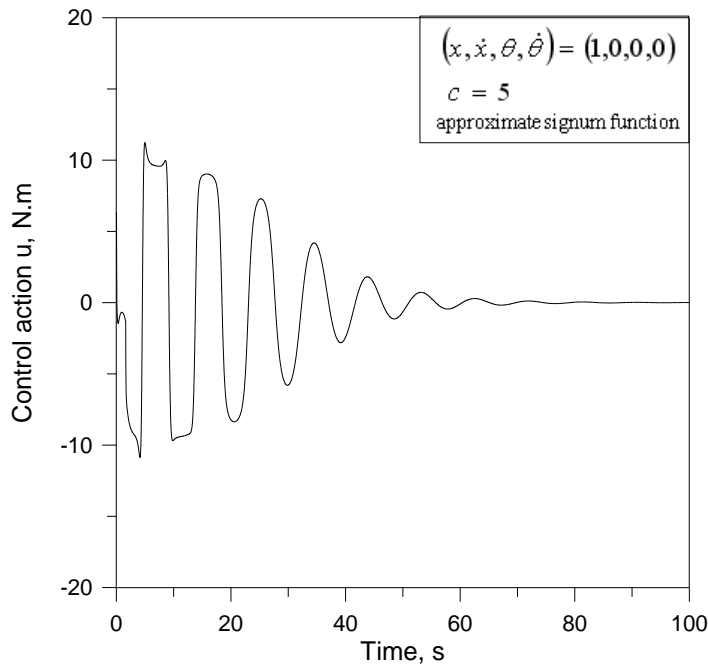**Fig. 11: Switching function versus time.**

Fig. 12: Control action versus time.

The uncertainty of the switching function (proposition (5.5), (Hameed 2007) (section (3.2))) is also tested. Figures 13 to 15 summarize the main results. Figure 13 shows the plot of displacement x with time while Figs. 14 and 15 show the plot of the switching function s and the controller u with time, respectively. The results shown in Figs. 13, 14, and 15 show that the system is AS in spite of the presence of the uncertainty in the switching function (the nominal parameter values is used in the calculation of switching function). On the other hand, the boundary layer does not appear clearly in the plot of the switching function in Fig. 14 due to the small uncertainty assumption in system parameters ($\pm 5\%$).
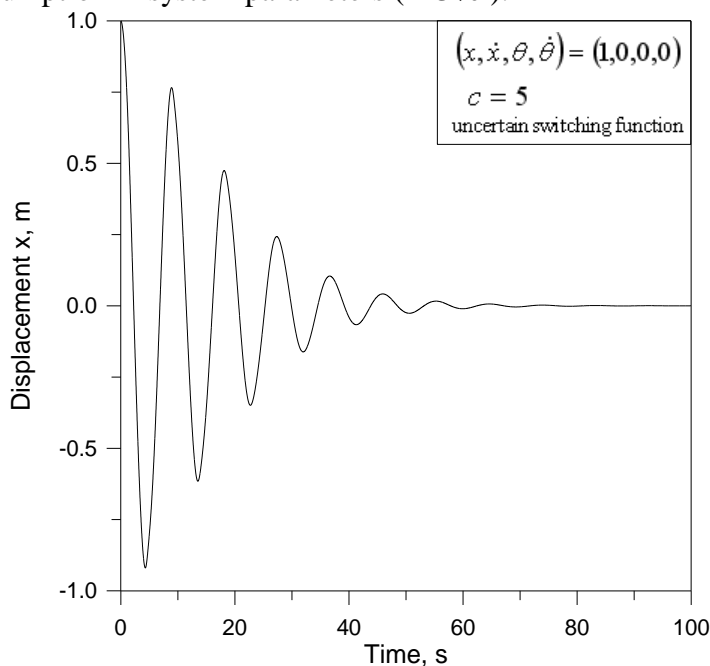

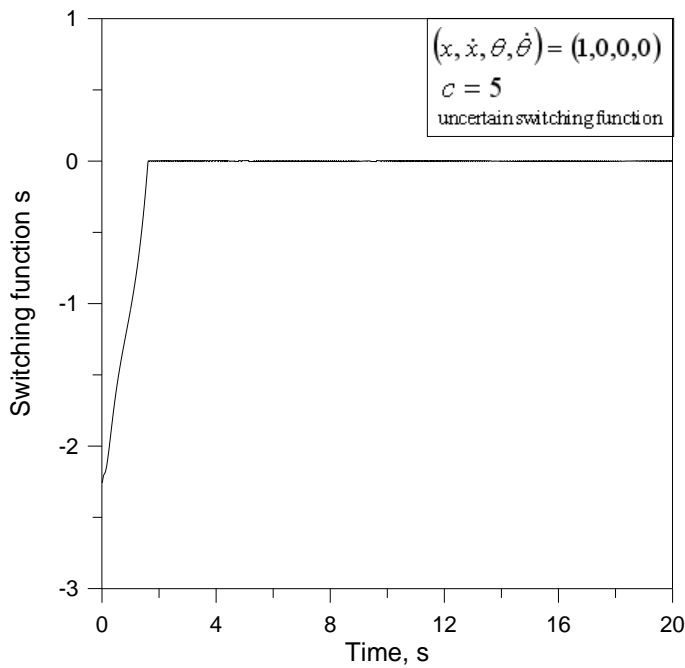**Fig. 13: Displacement versus time.**

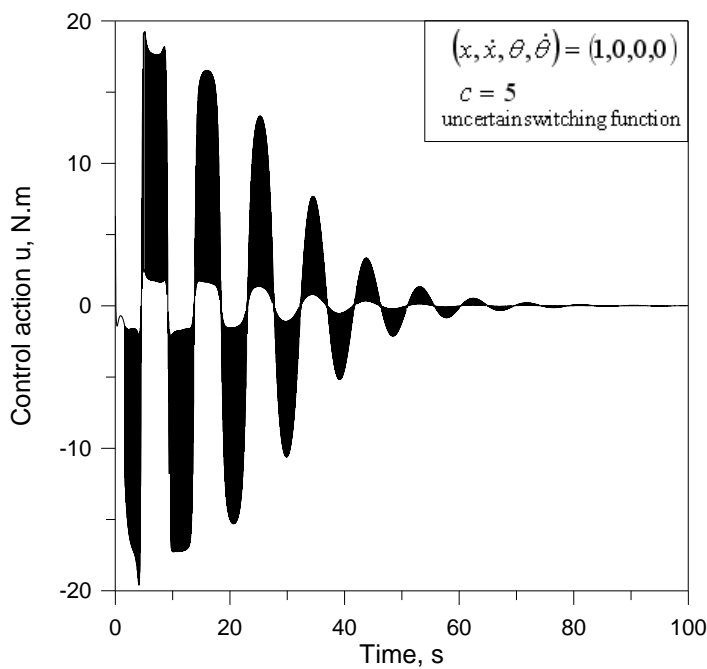**Fig. 14: Switching function versus time.**

Fig. 15: Control action versus time.

CONCLUSIONS

The present work shows the effectiveness of the sliding mode control theory for the design of a nonlinear controller for the TORA system as a two DOF underactuated mechanical system with model uncertainty. When the sliding controller force the state to the sliding manifold, the TORA system behaves like a spring–mass system with nonlinear damping effect due to the actuation of the eccentric mass. A numerical simulation shows that the proposed controller is robust with respect to the disturbances and uncertainty in system model. In addition the sliding mode controller is found

5601

effective in spite of the uncertainty in the switching function. Finally, these simulations prove the applicability of the new sliding controller formula in equations (28) and (29).

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NOMENCLATURE

<u>Symbols</u>

AS: Asymptotically Stable.
DOF: Degree Of Freedom.
GAS: Globally Asymptotically Stable.
ISS: Input-to-State Stable.
g_o: Acceleration due to gravitational attraction ($g_0 \approx 9.8\, m/s^2$).

I : Moment of inertia of the eccentric mass about its own center.

K : Spring constant [N/m].

$L_f s, L_g s$: Lie derivative of switching function with respect to f and g vector felids respectively.

$(L_f s)_{cer}, (L_f s)_{unc}$: Certain and uncertain parts of $L_f s$

m_1 : Platform mass [Kg].

m_2 : Eccentric mass [Kg].

r : Rotor radius [m].

sgn() : The signum function.

$sgn( )_{app}$: Approximate signum function.

GREEK SYMBOLS

Δ : Determinant of the inertia matrix ($\Delta = \det M$).

δ_f , δ_g : percent estimation of the variation of $L_f s$ and $L_g s$ from their nominal values, respectively.

λ, c : controller design parameters

APPENDIX A

Computing Controller Formula For The Tora System

To compute the control action u for the TORA system the rate of change of the switching function is computed first as:

$$e = \theta + \tan^{-1}(cy_2) = \theta + \tan^{-1}(\eta) \tag{A.1}$$

$$\dot{e} = \dot{\theta} + \frac{\dot{\eta}}{1+(\eta)^2} \tag{A.2}$$

$$\ddot{e} = \ddot{\theta} + \frac{\ddot{\eta}}{1+(\eta)^2} - \frac{2(\eta)(\dot{\eta})^2}{[1+(\eta)^2]^2} \tag{A.3}$$

where $\eta = cy_2 = c\dot{y}$ then;

$$\dot{s} = \ddot{\theta} + \frac{\ddot{\eta}}{1+(\eta)^2} - \frac{2\eta(\dot{\eta})^2}{[1+(\eta)^2]^2} + \dot{\theta} + \frac{\dot{\eta}}{1+(\eta)^2} \tag{A.4}$$

In addition $L_f s$ and $L_g s$ are given by;

$$L_f s = -\frac{(m_2 r)^2 \sin(\theta)\cos(\theta)}{\Delta}(\dot{\theta})^2 + \frac{(m_2 rK)\cos(\theta)x}{\Delta} +$$
$$\frac{(m_1 + m_2)m_2 g_0 r \sin(\theta)}{\Delta} + \frac{\ddot{\eta}}{1+(\eta)^2} - \frac{2\eta(\dot{\eta})^2}{[1+(\eta)^2]^2} + \dot{\theta} + \frac{\dot{\eta}}{1+(\eta)^2} \tag{A.5}$$

$$L_g s = \left(\frac{m_1 + m_2}{\Delta}\right) \tag{A.6}$$

Following eq. (21), $L_f s$ is decomposed to the following form;

$$L_f s = \sum_{i=1}^{3} \left(L_f s \right)_{unc,i} + \left(L_f s \right)_{cer} \tag{A.7}$$

where,

$$\left(L_f s \right)_{cer} = \frac{\ddot{\eta}}{1+(\eta)^2} - \frac{2\eta(\dot{\eta})^2}{\left[1+(\eta)^2\right]^2} + \dot{\theta} + \frac{\dot{\eta}}{1+(\eta)^2} \tag{A.8}$$

$$\left(L_f s \right)_{unc,1} = -\frac{(m_2 r)^2}{\Delta} \sin(\theta)\cos(\theta)(\dot{\theta})^2 \tag{A.9}$$

$$\left(L_f s \right)_{unc,2} = \frac{(m_2 r K)}{\Delta} \cos(\theta)x \tag{A.10}$$

$$\left(L_f s \right)_{unc,3} = \frac{(m_1 + m_2)m_2 g_0 r}{\Delta} \sin(\theta) \tag{A.11}$$

The nominal parameter values[1] are $m_1 = 10$, $m_2 = r = I_2 = 1$, $K = 5$ (Olfati-Saber 2001), $g_0 = 9.81$. Assuming the maximum/minimum variation values are bounded by $\pm 5\%$ of its nominal values, then $\left(\left(L_f s \right)_{unc,i} \right)_{nom}$, $\left(L_f s \right)_{nom}$ and $\left(L_g s \right)_{nom}$ are;

$$\left(\left(L_f s \right)_{unc,1} \right)_{nom} = -\left[\frac{1}{21+(\sin(\theta))^2} \right] \sin(\theta)\cos(\theta)(\dot{\theta})^2 \tag{A.12}$$

$$\left(\left(L_f s \right)_{unc,2} \right)_{nom} = \left[\frac{5}{21+(\sin(\theta))^2} \right] \cos(\theta)x \tag{A.13}$$

$$\left(\left(L_f s \right)_{unc,3} \right)_{nom} = \left[\frac{110}{21+(\sin(\theta))^2} \right] \sin(\theta) \tag{A.14}$$

$$\left(L_f s \right)_{nom} = \frac{-\sin(\theta)\cos(\theta)(\dot{\theta})^2 + 5\cos(\theta)x + 110\sin(\theta)}{21+(\sin(\theta))^2} + \frac{\ddot{\eta}}{1+(\eta)^2} - \frac{2\eta(\dot{\eta})^2}{\left[1+(\eta)^2\right]^2} + \dot{\theta} + \frac{\dot{\eta}}{1+(\eta)^2} \tag{A.15}$$

$$\left(L_g s \right)_{nom} = \frac{11}{21+(\sin(\theta))^2} \tag{A.16}$$

For $\left| \overline{\left(L_f s \right)_{nom}} \right|$, compute first δ_i, $i = 1,2,3$ of the bounded estimation for each component of $\left(L_f s \right)_{unc}$ with $\min \Delta = 18.07 + 0.81(\sin(\theta))^2$ as in the following;

$$\left| \left(L_f s \right)_{unc,1} - \left(\left(L_f s \right)_{unc,1} \right)_{nom} \right| < \left| \frac{(1.05)^4}{18.07+0.81(\sin(\theta))^2} - \frac{1}{21+(\sin(\theta))^2} \right| \sin(\theta)\cos(\theta)(\dot{\theta})^2$$

$$= \left| \frac{\left((1.05)^4 * 21 - 18.07\right) + \left((1.05)^4 - 0.81\right)(\sin(\theta))^2}{18.07+0.81(\sin(\theta))^2} \right| \left| \left(\frac{\sin(\theta)\cos(\theta)(\dot{\theta})^2}{21+(\sin(\theta))^2} \right) \right|$$

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[1] Throughout the simulations in this work the units of physical quantities is taken as meter unit for length quantity, kilogram for mass, and Newton for force.

However, $\max\left|\dfrac{\left((1.05)^4 * 21 - 18.07\right) + \left((1.05)^4 - 0.81\right)(\sin(\theta))^2}{18.07 + 0.81(\sin(\theta))^2}\right| < 0.42 = \delta_1$ at $\sin(\theta) = 1$. Then we

have:

$$\left|(L_f s)_{unc,1} - \left((L_f s)_{unc,1}\right)_{nom}\right| < 0.42\left|\left((L_f s)_{unc,1}\right)_{nom}\right| \tag{A.17}$$

In the same way, we get;

$$\left|(L_f s)_{unc,2} - \left((L_f s)_{unc,2}\right)_{nom}\right| < 0.29\left|\left((L_f s)_{unc,2}\right)_{nom}\right| \tag{A.18}$$

$$\left|(L_f s)_{unc,3} - \left((L_f s)_{unc,3}\right)_{nom}\right| < 0.35\left|\left((L_f s)_{unc,3}\right)_{nom}\right| \tag{A.19}$$

The term $\left|\overline{L_f s}\right|$ takes the following form;

$$\left|\overline{L_f s}\right|_{nom} = \sum_{i=1}^{3}\frac{\delta_i}{\delta_f}\left|\left((L_f s)_{unc,i}\right)_{nom}\right| + \left|(L_f s)_{cer}\right| = \left(\frac{0.42}{0.29}\right)\left|\left(\overline{L_f s_1}\right)_{nom}\right|$$

$$+ \left|\left(\overline{L_f s_2}\right)_{nom}\right| + \left(\frac{0.35}{0.29}\right)\left|\left(\overline{L_f s_3}\right)_{nom}\right| + \left|\frac{c\ddot{y}}{1+(c\dot{y})^2} - \frac{2c^3\dot{y}(\ddot{y})^2}{\left[1+(c\dot{y})^2\right]^2} + \dot{\theta} + \frac{c\ddot{y}}{1+(c\dot{y})^2}\right| \tag{A.20}$$

where $\delta_f = \min[\delta_1, \delta_2, \delta_3] = \min[0.42, 0.29, 0.35] = 0.29$.

Also for $L_g s$, δ_g is computed as;

$$\left|L_g s - (L_g s)_{nom}\right| < \left|\frac{(10.5+1.05)}{18.07+0.81(\sin(\theta))^2} - \frac{11}{21+(\sin(\theta))^2}\right| < \delta_g(L_g s)_{nom} = 0.23(L_g s)_{nom} \tag{A.21}$$

Now the sliding controller as in proposition (5.2), (Hameed 2007) is given by:

$$u = -\left(\frac{L_f s}{L_g s}\right)_{nom} - v_0 \operatorname{sgn} s \tag{A.22}$$

$$v_0 = 1.1\left[\frac{0.29+0.23}{1-0.23}\right]\left|\left(\frac{\overline{L_f s}}{L_g s}\right)_{nom}\right| = 0.68\left|\left(\frac{\overline{L_f s}}{L_g s}\right)_{nom}\right| \tag{A.23}$$