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FREE VIBRATION ANALYSIS OF TAPERED BOX GIRDER USING GRILLAGE ANALOGY AND FINITE ELEMENT METHODS.

Meethaq Sh. Al-Fatlawi
Assistant Inst
University of Baghdad

ABSTRACT

A general method using the theory of thin-walled structures is given for determining the natural frequencies and mode shape for box –girder of varying depth having closed section, subjected to torsion and bending using a grillage method and finite element method (plate /shell, MSC, NASTRAN) . Consistent mass matrices related to torsion and bi- moment effects have been derived using shape functions corresponding to an assumed polynomial deflection configuration, also stiffness and consistent mass matrices for flexural behavior including the effect of shear deformation and rotary inertia in bending. The stiffness matrix for beam element under non-uniform torsion is presented by using the differential equation of equilibrium.

A special computer program is written to perform the free vibration analysis starting from the element stiffness and consistent mass matrices. The results have been compared with those obtained from MSC /NASTRAN Package. Numerical examples are presented to show the effect of cell number and effect of variation for (span / width) ratio.

الخلاصة

تم استخدام طريقة عامة معتمدة على نظرية المنشآت اللوحية الرقيقة الجدران لإيجاد التردد الطبيعي و شكل الطور الخاص بالمنشآت اللوحية الخلوية المتغيرة العمق والمعرضة لتأثيري الالتواء حول المحور الطولي والانحناء.تم استخدام طريقتين للتحليل الأولى: باستخدام طريقة المشبكات المبسطة والطريقة الثانية باستخدام العناصر المحددة (الألواح القشرية الرقيقة) من خلال برنامج جاهز (MSC/NASTRAN).

تم إيجاد مصفوفات الكتل المتوافقة المتعلقة بتأثير الالتواء حول المحور الطولي والعزم الإضافي نتيجة تقييد تأثير ألالتواء للمقطع , كذلك مصفوفات الكتل المتوافقة تحت تأثير الانحناء مع إدخال تأثير التشوه القصي والقصور الذاتي للدوران , و إيجاد مصفوفات الصلادة لكل عنصر مكافئ تحت تأثير الالتواء غير المنتظم والانحناء.

أعد برنامج حسابي لإنجاز تحليل الاهتزاز الحر ابتدأ من تجميع مصفوفات الصلادة والكتلة لكل عنصر مكافئ من المنشأ وتم مقارنة النتائج مع تلك المستحصلة من البرنامج الجاهز. تم إعداد أمثلة لغرض دراسة تأثير عدد الخلايا للمقطع العرضي ودراسة تأثير نسبة (الطول/العرض) على التردد الطبيعي.

KEY WORDS
Free vibration, tapered box girder, grillage, finite element, thin-walled structures.

INTRODUCTION

Plane girder can be idealized as structural systems formed by unidimensional finite elements. These elements are the bars whose assemblage forms the structure.

The Candian High Way Bridge Design Code (CHBDC 2000) as well as the American Association of Flat High Way Transportation Officials (AASHTO 1996; AASHTO 1994) has recommended several method of analysis for only straight box- girder bridges. These methods include orthotropic plate theory, finite deference technique, grillage analogy, folded plate, finite strip and finite element technique.

The grillage analogy method is applied to the free vibration analysis of grids with linearly varying depth **Fig.(1)**. An engineering theory of torsion bending has been developed for non-uniform beam with multi-cell cross section. The distribution of warping restraint stresses around the section is defined in a similar way as for bending by a system of sectional co-ordinates and several additional geometrical terms.

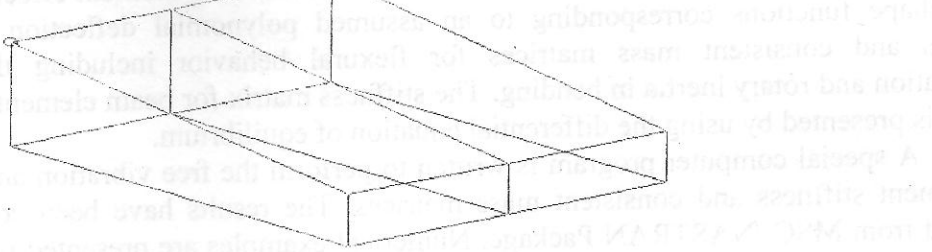


Fig.(1) Tapered Box Girder

MULTI- CELLULAR SECTIONS

Box girders with more than one cell are frequently found in practice and enable the range of application of cellular structures to be greatly extended. Their behavior is not significantly different from that of single cell box girders in which a shear flow of a constant magnitude develops around the box to resist pure applied torque.

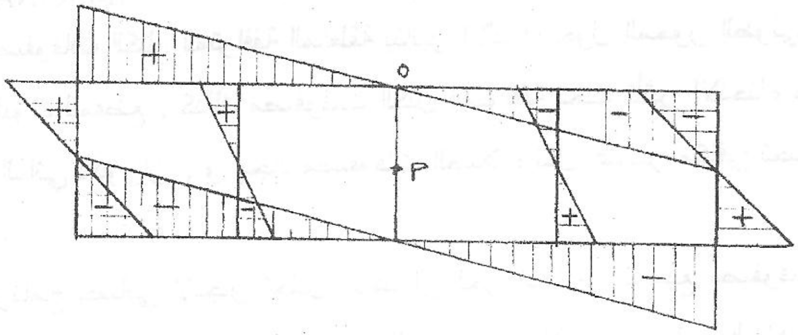


Fig.(2). Determination of the sectorial area (ω) for uniform multi-cell closed section.

The co-ordinate distribution represents the level of out of plane warping due to a unit rate of twist and must accordingly have zero value at points on the cross -section which display zero warping. If the position of shear center and a point of zero warping are not obvious from symmetry, they may be determined by satisfying the conditions [Waldron, 1986]

$$\int_A \omega_{(s)}.dA = \int_A \omega_{(s)}.x.dA = \int_A \omega_{(s)}.y.dA = 0 \qquad (1)$$

The sectional co-ordinates for a closed section are determined by first introducing an imaginary cut to the closed cell there by transforming it into an equivalent open section. Account must then be taken of the connectivity condition for the closed cell in which it is necessary to restore compatibility at the imaginary cut . This leads to the following expression for unit warping.

$$\omega_{(s)} = \int_0^s r_t.ds - \frac{\Omega}{\oint \frac{ds}{t}} \int_0^s ds \qquad (2)$$

in which the first integral refers to the equivalent open section and is therefore evaluated around the entire section . The second integral applies only to the closed part of the section effectively reducing warping displacements by restoring continuity around the cell .
The stress resultant (Bi) and (Tw) are defined as

$$Bi = -EI_w \xi''$$
$$Tw = Bi' = -EI_w \xi''' \qquad (3)$$

and are obtained from a solution to the partial differential equation for non- uniform torsion:

$$T = GJ\phi' - EI_w \xi''' \qquad (4)$$

The deformation term (ξ) is a non –dimensional warping function closely related to the twist ϕ by the expression:

$$\phi' = \mu_\omega \xi + \frac{T}{GIc} \qquad (5)$$

Where (Ic) is the central second moment of area and μ_ω is warping shear parameter , given by.[Benscoter , 1954].

$$Ic = \int_s r_t.dA$$

$$\mu_\omega = 1 - \frac{J}{Ic} \qquad (6)$$

Torsional Constant for closed Cell Sections.

A procedure to calculate the torsional constant (J) for multi-cell cross section under pure torsion and free to warp is given by [Timoshinko and Goodier , 1970]

$$\Omega_i = \left[\psi_i \int_i \frac{ds}{t} - \sum_{k=1}^{m} \psi_k \int_k \frac{ds}{t} \right] \quad (7)$$

Where (ψ_i) is the geometric quantity and is easily determined by the middle line of the section for given cell (i):

$$\psi_i = \frac{q}{G.\,{}^{d\phi}/_{dx}} \quad (8)$$

So, $J_i = \psi_i \Omega_i$ (9)

CONSISTENT MASS MATRIX OF BEAM ELEMENT UNDER

NON- UNIFORM TORSION

Two degree of freedom ϕ and $\theta = \frac{d\phi}{dx}$ have been allowed at each node point of a beam element subjected to non- uniform torsion. These parameters are analogous to that of flexural element with two degree of freedom w and $\frac{dw}{dx}$ at each node. Based on this analogy the displacement configuration associated with each degree of freedom ϕ and can be represented by a cubic polynomial as described below [Mallick and Dungar , 1977].

$$\begin{aligned}
\phi_{1(z)} &= -z + \left(2z^2/L\right) - \left(z^3/L^2\right) \\
\phi_{2(z)} &= 1 + 2\left(2z^3/L^3\right) - \left(3z^2/L^2\right) \\
\phi_{3(z)} &= \left(z^2/L\right) - \left(z^3/L^2\right) \\
\phi_{4(z)} &= \left(3z^2/L^2\right) - \left(2z^3/L^3\right)
\end{aligned} \quad (10)$$

Where $\phi_1(z), \phi_2(z), \phi_3(z)$ and $\phi_4(z)$ are the shape functions associated with each degree of freedom ϕ_1, θ_1, ϕ_2 and θ_2.

Using the shape functions, the complete mass matrix for beam element can be derived using the above shape functions.

Because of the fact that the acceleration distribution follows the displacement distribution pattern during sinusoidal vibrations, it is now possible to define the inertia load ,function f(x) , corresponding to a unit value of acceleration ϕ_1

$$f_{1(x)} = m(C_1 + D_1 Z + E_1 \text{Sinhaz} + F_1 \text{Coshaz}) \quad (11)$$

where (m) is the mass moment of inertia per unit length of the element and its equal to $I_j.\frac{\bar{\gamma}}{g}$ where $\bar{\gamma}$ is the weight density per unit length of the member and I_j is the polar moment of inertia of the cross section about the centroidal longitudinal axes. The generalized mass (m_{ij}) can be written as :

$$m_{ij} = m\int_0^L f_{i(z)}.\phi_{j(z)}.dz \tag{12}$$

After integration , the mass matrix (4×4) of thin-walled beam element with open or closed section subjected to degree of freedom ϕ and θ at each node can be complied will be of the form . [Mallick , 1977].

$$[m] = I_j.\frac{\bar{\gamma}}{g}\begin{bmatrix} \frac{13L}{35} & & & \\ \frac{11L^2}{210} & \frac{L^3}{105} & & \\ \frac{9L}{70} & \frac{13L^2}{420} & \frac{13L}{35} & \\ \frac{-13L^2}{420} & \frac{-L^2}{140} & \frac{-11L^2}{210} & \frac{L^3}{105} \end{bmatrix} \tag{13}$$

For tapered members, $I_j = \bar{I}_j$ where $\bar{I}_j$ is the mean value of polar moment of inertia of the cross section at the two ends.

It is possible to improve the procedure by considering the inertial forces caused by rotary inertia [Venancio, 1973 and Briseghella 1998]. For this purpose it is sufficient to add the following matrix to the mass matrix.

$$M_{rote} = \frac{\bar{\gamma}}{g}.I_{\omega}.\begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{15}L & \frac{-1}{10} & \frac{-1}{30}L \\ \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\ \frac{1}{10} & \frac{-1}{30}L & \frac{-1}{10} & \frac{2}{15}L \end{bmatrix} \tag{14}$$

I_{ω} = warping constant of the cross section of beam.

Element Mass Matrix Under Bending

It is possible to obtain mass matrix, [Bresighella, 1998].

$$m_{ij} = \int_0^L N_i.m.N_j.dx$$

$$M_e = \frac{\bar{\gamma}}{g}.\frac{L}{630}.\begin{bmatrix} 234 & 33L & 8L & -9.5L \\ 33L & 6L^2 & 19.5L & -4.5L^2 \\ 81 & 19.5L & 234 & -33L \\ -19.5L & -4.5L^2 & -33L & 6L^2 \end{bmatrix} \tag{15}$$

TORSIONAL STIFFNESS MATRIX FOR THIN-WALLED BEAM ELELMENT

The solution of the differential Equation, [Just and Walley, 1979]

$$\phi^{iv} - k^2 . \phi'' = -\frac{\mu_\omega . m_t}{E_1 . I_\omega} - \frac{m_t''}{G . I_c} \tag{16}$$

Can be assumed to be:

$$\phi = C_1 + C_2 x + C_3 \sinh(kx) + C_4 \cosh(kx) \tag{17}$$

$$\text{or} \quad \phi(x) = [1 \quad x \quad \sinh(kx) \quad \cosh(kx)]\ \{\alpha\} \tag{18}$$

Where here $\{C1, C_2, C_3, C_4\}^T$.

$$\theta(x) = \left[0 \quad 1 \quad \frac{k}{\mu_\omega}\cosh(kx) \quad \frac{k}{\mu_\omega}\sinh(kx)\right]\{\alpha\} \tag{19}$$

The boundary displacements are given by:

$$[\Phi] = [A]\ \{\alpha\}$$

or in a matrix form:

$$\begin{Bmatrix} \phi_1 \\ \theta_1 \\ \phi_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{k}{\mu_w} & 0 \\ 0 & L & S & C \\ 0 & 1 & \frac{k}{\mu_w}C & \frac{k}{\mu_w}S \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \tag{20}$$

Hence;

$$[K_\phi] = [B]\ [A]^{-1}$$

So, finally:

$$[K_\phi] = \overline{GJ} \begin{bmatrix} k\,SH & \mu_\omega(1-CH) & -k\,SH & \mu_\omega(1-CH) \\ & \frac{\mu_\omega}{k}(kLCH-\mu_\omega SH) & -\mu_\omega(1-CH) & \frac{\mu_\omega}{k}(\mu_\omega SH-k) \\ & & k\,SH & -\mu_\omega(1-CH) \\ & & & \frac{\mu_\omega}{k}(kLCH-\mu_\omega SH) \end{bmatrix} \tag{21}$$

Where:

$$\overline{GJ} = \frac{GJ}{\left[2\mu_{\omega}(1-CH)+kL\ SH)\right]}$$

$J = J^{*}$ (where J^{*} is the mean value of torsional constant for varying depth beam element).

Stiffness Matrix For Prismatic (I- Beam) Element

$$K_c = \frac{EI_y.\beta}{1-v^2}.\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\beta)L^2 & -6L & (2-\beta)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\beta)L^2 & -6L & (4+\beta)L^2 \end{bmatrix} \quad (22)$$

$\beta = \frac{12EI_y}{GA_s L^2}$ (for I and box or cellular section member).

A_s is the area of the web plate for member of (I) section and box or cellular section.

The effective flange width due to shear lag is considered by using the table of [Moffat and Doweling , 1975].

Stiffness Matrix For Tapered (I-Beam) Element

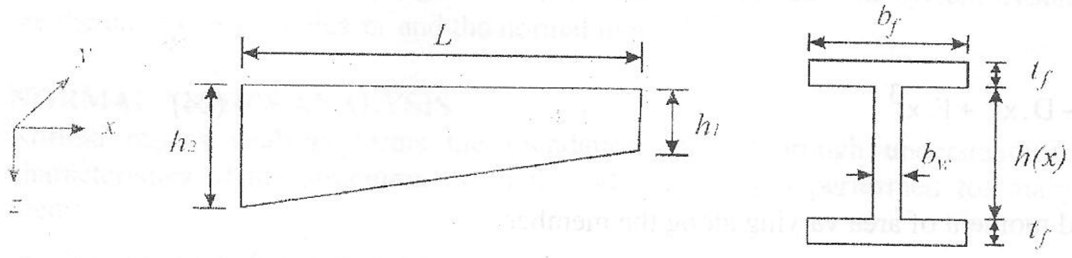


Fig. (3) I-cross section of a taper grillage member

$$\begin{bmatrix} V_1 \\ M_{y1} \\ V_2 \\ M_{y2} \end{bmatrix} = \begin{bmatrix} a & b & -a & c \\ & d & -b & e \\ & & a & -c \\ & & & n \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{bmatrix} \quad (23)$$

In the above. [Meethaq , 2004]

$$a = \frac{1}{BB} \; ; \quad b = \frac{\left(ZZ_1/ZZ_3\right)}{BB} \; ; \quad c = \frac{\left(L - \left(ZZ_1/ZZ_3\right)\right)}{BB} \; ; \quad d = \frac{\left(ZZ_1/ZZ_3\right)^2}{BB} + \frac{1}{ZZ_3}$$

$$e = \frac{\left(ZZ_1/ZZ_3\right)\left(L - \left(ZZ_1/ZZ_3\right)\right)}{BB} - \frac{1}{ZZ_3} \; ; \quad n = \frac{\left(L - \left(ZZ_1/ZZ_3\right)\right)^2}{BB} + \frac{1}{ZZ_3}$$

where

$$ZZ_1 = \int_0^L \frac{x}{EI_y^*}\,dx \; ; \quad ZZ_2 = \int_0^L \frac{x^2}{EI_y^*}\,dx \; ; \quad ZZ_3 = \int_0^L \frac{dx}{EI_y^*} \; ; \quad ZZ_4 = \int_0^L \frac{dx}{GA_s^*}$$

$$BB = ZZ_2 + ZZ_4 - \frac{ZZ_1^2}{ZZ_3}$$

$$B = \frac{b_w\, h_1^3}{12} + \frac{1}{(1-v^2)}\left[\frac{2}{3}\, b_f\, t_f^3 + \frac{1}{2}\, b_f\, t_f\, h_1^2 + b_f\, t_f^2\, h_1\right]$$

$$C = \left[\frac{b_w}{4}\, h_1^2 + \frac{1}{(1-v^2)}\left(b_f\, t_f\, h_1 + b_f\, t_f^2\right)\right].\eta$$

$$D = \left[\frac{b_w}{4}\, h_1 + \frac{1}{2(1-v^2)}\, b_f\, t_f\right].\eta^2 \; ; \quad F = \frac{b_w}{12}\, \eta^3$$

Thus:

$$I_y^*(x) = B + C.x + D.x^2 + F.x^3 \qquad (24)$$

I_y^*: is the second moment of area varying along the member.

A_s^* : is the area of shear-carrying web plate varying along the member.

Total Stiffness Matrix For Each Beam Element

Matrix of (8×8) represent the stiffness of each element, which is involved the flexural and torsional matrices as shown below.

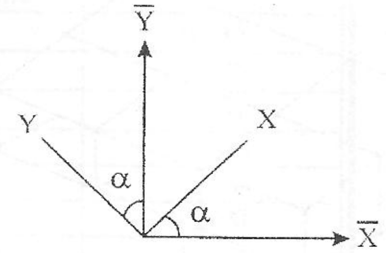
$$K_T = \begin{bmatrix} K_e & \\ & K_\phi \end{bmatrix} \qquad (25)$$

GRID MATRICES AND DYNAMIC CHARACTERISTICS

The element stiffness and consistent mass matrices in element coordinates are transformed to system coordinates through the standard transformations.

$$[\overline{K}_T] = [H]^t [K_T][H] \tag{26}$$

where



$$[H] = \begin{bmatrix} [h] & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [h] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

With the element matrices $[k_T]$ and $[M_e]$ the system matrices $[\overline{K}_T]$ and $[M]$ are built up.

The free vibration of a structural system is governed by the equation.

$$[M]\{\ddot{P}\} + [\overline{K}_T]\{P\} = 0 \tag{27}$$

For free vibrations in normal mode. $\{P\} = \{P_o\}.\sin \varpi t$

Substituting this in equation (27) , one obtains

$$\varpi^2 [M]\{P_o\} = [\overline{K}_T]\{P_o\} \tag{28}$$

Equation (28) represents the eign value problem to be solved. The system dynamic characteristics are the natural frequencies ϖ and the normal mode $\{P_o\}$.

NORMAL MODES ANALYSIS

Normal modes analysis forms the foundation for a thorough understanding of the dynamic characteristics of the structures. Normal modes analysis is performed for many reasons, among them:

- Assessing the dynamic interaction between a component (such as a piece of rotating machinery) and its supporting structures is close to an operating frequency of the component, and then there can be significant dynamic amplification of the loads.
- Assessing the effect of design changes on the dynamic characteristics and used for another reasons.

<u>Example(1)</u>

The natural frequencies of tapered box girder of **Fig.(4)** and **Fig.(5)** are obtained by using two methods : first grillage analogy method including warping effect , second finite element method (Flat Plate /Shell) from MSC/ NASTRAN Packages

Three different sections of tapered box girder are discussed. The first with one

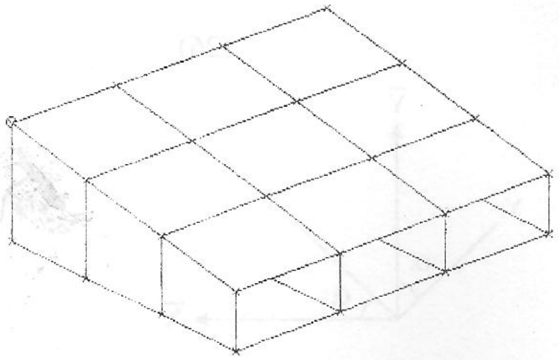


Fig. (5) Tapered Box Girder with Two Intermediate Stiffeners.

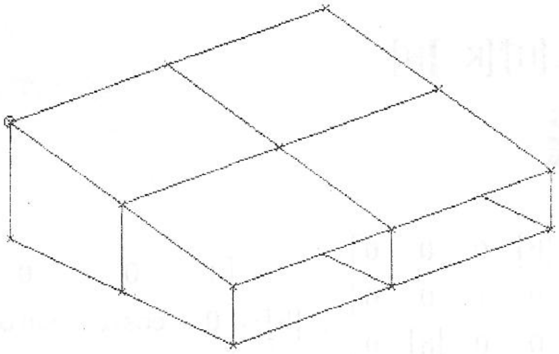


Fig. (4) Tapered Box Girder with One Intermediate Stiffeners

Intermediate diaphragms, as shown in **Fig. (4)** . The other type with two intermediate diaphragms. **Fig. (5)**. the last one with three intermediate stiffeners.
Type of mesh which is used in (MSC/NASTRAN Packages) is presented in **Fig (6)**. The box girder constrained from one edges (Cantilever).

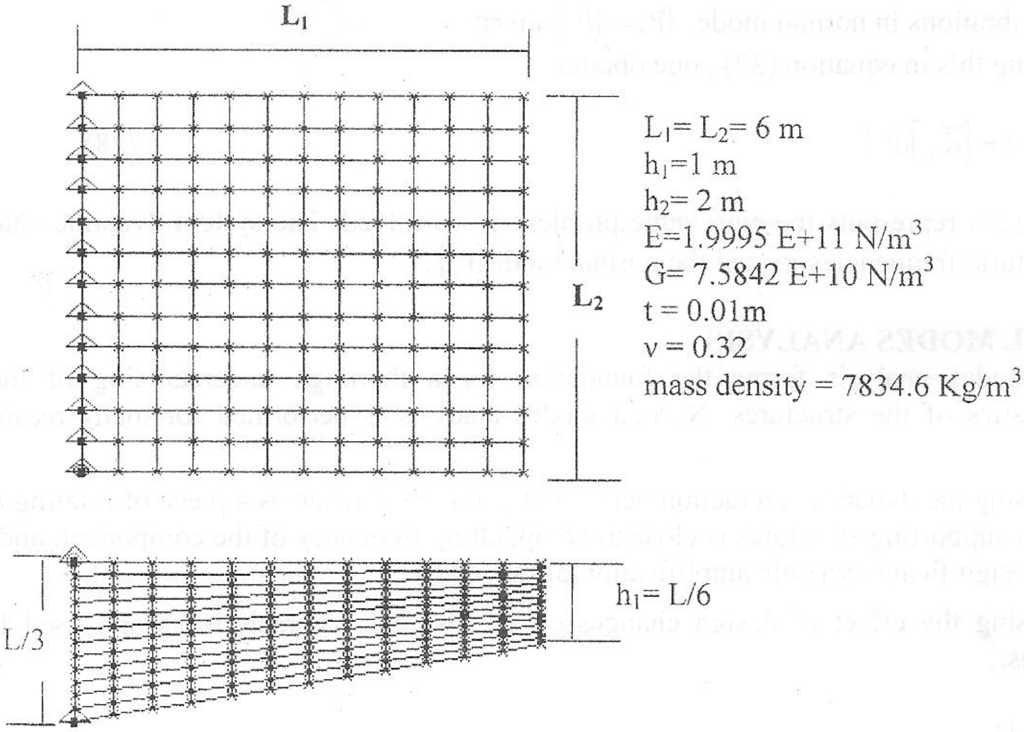


Fig.(6). Finite Element Mesh

The resulting natural frequencies are presented in **Table. (1)**.

Table. (1). Natural Frequencies ϖ (Hz) for Cantilever Cellular Plate Structure (Tapered)

Cell No.	Mode	Grillage Method Including Warping Effect	Finite Element MSC/NASTRAN
2	1	6.425	6.815
	2	6.945	7.625
	3	7.541	8.181
3	1	13.510	14.241
	2	14.721	15.031
	3	15.546	15.358
4	1	19.203	20.920
	2	18.910	21.386
	3	20.215	22.861

For the lower modes the agreement between the grillage analogy method and finite element solution is good. **Fig.(7)** show the effect of cell number and position of stiffener on natural frequency with this type of supporting.

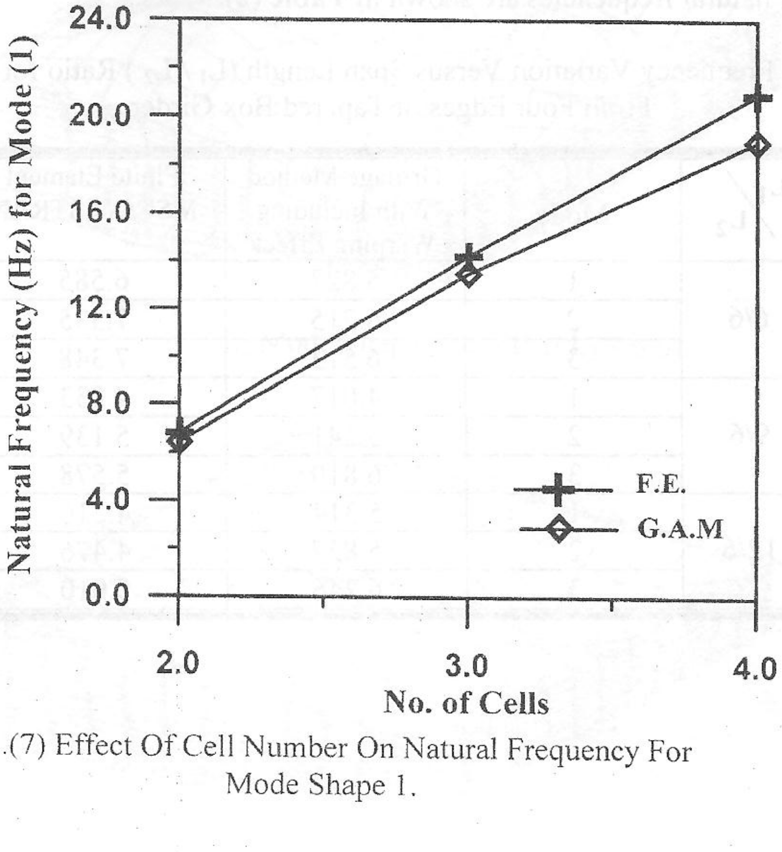


Fig .(7) Effect Of Cell Number On Natural Frequency For Mode Shape 1.

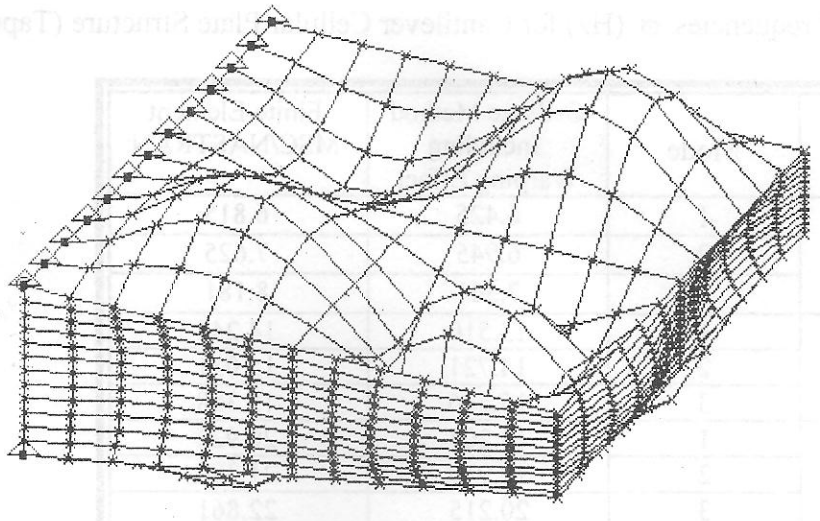


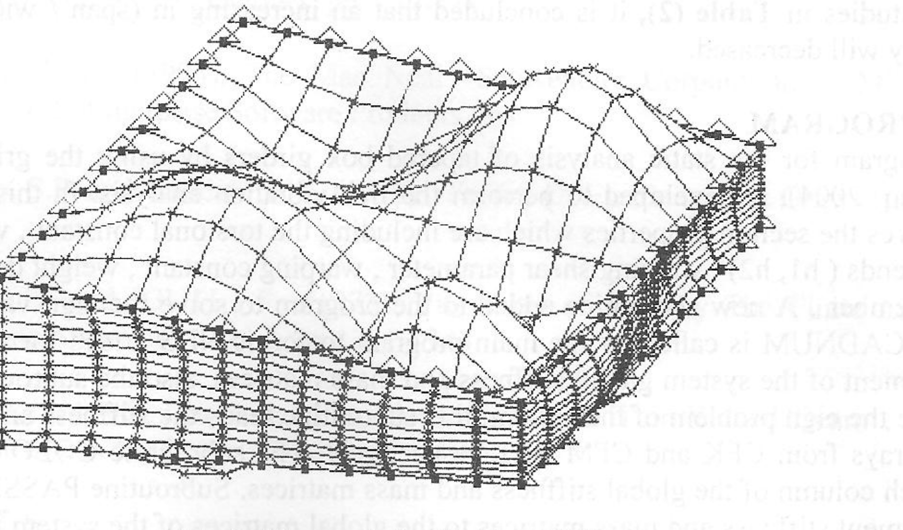
Fig.(8) Free Vibration for Mode Shape (1) .

Example (2)

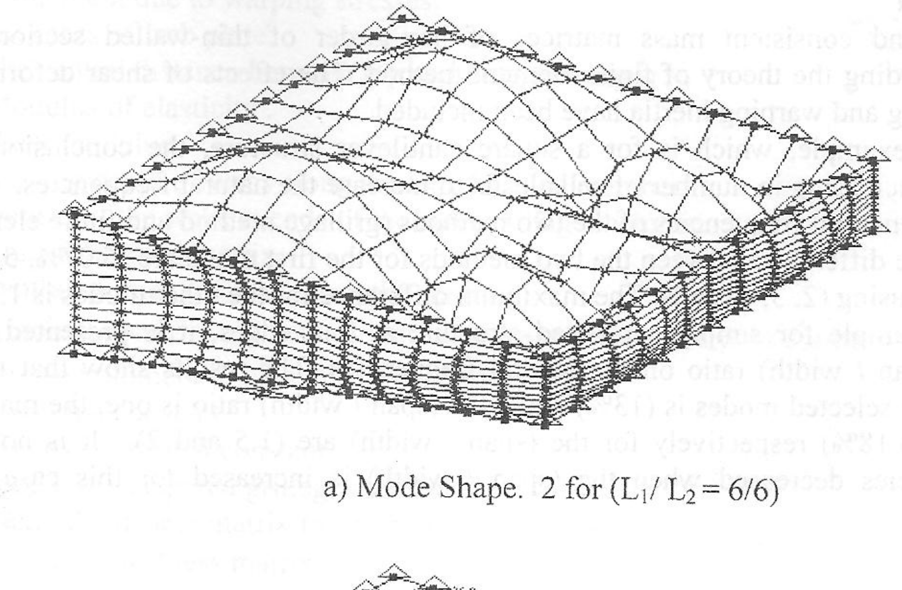
A simply supported from four edges have the same material property of example (1) is considered to study the effect of variation of (span / width) ratio, the cross section of the structure have two cells. The results of natural frequencies are shown in **Table (2).**

Table (2). Natural Frequency Variation Versus Span Length (L_1 / L_2) Ratio for Simply Supported From Four Edges of Tapered Box Girder.

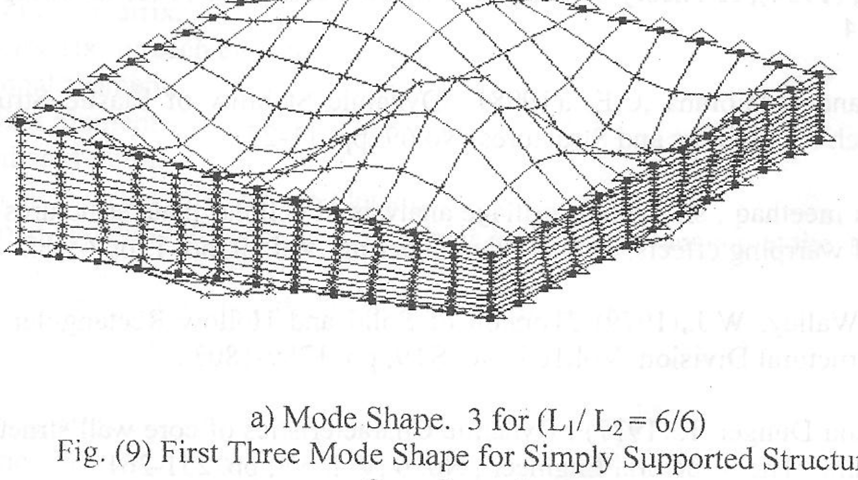
L_1/L_2	Mode	Grillage Method With Including Warping Effect	Finite Element MSC/NASTRAN
6/6	1	5.827	6.585
	2	6.215	7.335
	3	6.513	7.348
9/6	1	4.017	4.833
	2	5.241	5.139
	3	6.810	5.578
12/6	1	5.314	4.216
	2	5.827	4.476
	3	6.246	5.010



a) Mode Shape. 1 for ($L_1/L_2 = 6/6$)



a) Mode Shape. 2 for ($L_1/L_2 = 6/6$)



a) Mode Shape. 3 for ($L_1/L_2 = 6/6$)

Fig. (9) First Three Mode Shape for Simply Supported Structures from four edges.

From the case studies in **Table (2),** it is concluded that an increasing in (span / width) ratio, the natural frequency will decreased.

COMPUTER PROGRAM

A computer program for the static analysis of tapered box girders by using the grillage method (Husain, meethaq, 2004), is developed to perform the free vibration analysis. In this program the input data involves the section properties which are including the torsional constant , width of beam . depths at each ends (h1, h2) , warping shear parameter , warping constant , weight density per unit length of the member . A new subroutine added to the program to solve the eign value problems. The subroutine CADNUM is called by the main program to forms NDS arrays for the number of the diagonal element of the system global stiffness and mass matrices, also the subroutine SSPACE is called to solve the eign problem of the system; this subroutine reads the stiffness and mass as one -dimensional arrays from CFK and CFM files, respectively. The subroutine COLOMH computes the height of each column of the global stiffness and mass matrices. Subroutine PASSEM is used to assemble the element stiffness and mass matrices to the global matrices of the system in the form of on – dimensional arrays SK and SM respectively.

CONCLUSION

The stiffness and consistent mass matrices of box-girder of thin-walled sections have been developed according the theory of finite element method. The effects of shear deformation, rotary inertia in bending and warping inertia have been included.

From the first example, which is for a square cantilever structure, the conclusion that can be obtained is an increasing in number of cells leads to increase the natural frequencies. A comparison for the resulting natural frequencies of the two methods (grillage method and finite element method) is presented. The difference between the two methods for the first mode are (6.07%, 5.4 %, 8.94 %) respectively for using (2, 3, 4) cells. The maximum difference for the other modes is 13%.

The second example for simply supported structure at four edges, it is presented the effect of variation of (span / width) ratio on the natural frequencies. The results show that the maximum difference in the selected modes is (13%) when the (span / width) ratio is one, the max. differences are (20.3% - 23.18%) respectively for the (span / width) are (1.5 and 2). It is noticed that the natural frequencies decreased when the (span / width) is increased for this case of boundary conditions.

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NOTATIONS

As	Shear area of grillage member and equal to area of the web plate
bf	Width of flange plate .
Bi	Bi-moment due to warping stresses.
bw	thickness of web plate.
d	The central distance between top and bottom flanges .
E	Modulus of elasticity .
G	Shear modulus .
$h_{(x)}$	Varying depth of webs along the length of the member.
h_1,h_2	Depths of the webs at ends of the taper grillage member.
[H]	Transformation matrix.
I_C	Central moment of area about the centroid.
I_ω	Principal sectorial moment of inertia of the section or warping constant .
I_y	Second moment of area due to bending.
J	Torsional constant .
J_{eff}	Effective torsional constant.
L	Length of a tapered grillage member .
K_e	Flexural stiffness matrix for each element.
K_ϕ	Torsional stiffness matrix.
K_T	Total stiffness matrix.
M	Total mass matrix.
M_e	Mass matrix for each element.
M_t	Torsional moment. .
M_y	Bending moment .
m_t	Distributed torque .
q	Shear flow or the force per unit length .
r_t	The perpendicular distance from the shear center to the tangent at the section under consideration.
S	Curvilinear coordinate on the middle line of the section.
$S_\omega(s)$	Principal sectorial static moment .
T	Torque at any section.
Tw	warping stresses.
t_f	thickness of flange plate.
u(s)	Warping displacement
U	Strain energy.

V	Shear force
W	Vertical displacement .
X,Y,Z	Cartesian coordinate axes.

Greek Symbol.

Ω	Double area enclosed by the middle line of the section.
$\bar{\gamma}$	weight density per unit length of the member
θ	Warping function.
θ_y	Bending rotation.
μ_{ω}	Warping shear parameter.
$\xi_{(s)}$	Angular displacement.
σ	Normal stress.
σ_{ω}	Normal stress due to warping.
ϕ	Angle of twist
ϕ'	Real rate of twist.
ψ	Torsional function
ω	Sectorial area .
$\bar{\omega}$	Natural frequencies.