GRILLAGE ANALYSIS OF CELLULAR PLATE STRUCTURES OF VARYING DEPTH WITH INCLUSION OF WARPING EFFECTS.

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ABSTRACT
A static analysis of cellular plate structures of linearly varying depth is presented by using the simplified grillage method. The cellular plate structure in this study consists of top and bottom flange plates (non-parallel) separated by longitudinal webs having varying depth. A variational formulation of thin-walled beam element with linearly varying depth has been carried out for the derivation of the stiffness matrix of the grillage members for various warping restraint effects. The effect of warping restraint in stiffness is included by two methods, firstly; the implicit method (through the torsional constant) and secondly; by the explicit method (as independent warping degree of freedom). Two different approaches are used to derive the torsional stiffness matrix, namely:by the assumed displacement field and by the differential equation of equilibrium. The effect of secondary shear stresses from the warping restraint is also considered.
The varying cross section properties of the equivalent grillage members are derived from the properties of the actual cellular plate structure.
Several numerical examples and applications are presented . The results are compared with those obtained from the flat plate / shell elements of MSC/NASTRAN program. Acceptable agreement is obtained.

الخلاصة

التحليل بطريقة المشبكات للالواح الخلوية ذات العمق المتغير بادخال تأثير التشويه الالتوائي.

تم تقديم تحليل ستاتيكي مرن للالوح الخلوية المتغيرة العمق باستخدام طريقة المشبكات المبسطة (simplified grillage analogy) . تكون الالواح الخلوية من صفيحتين (عليا وسفلى) غير متوازيتين و مفصولتين بصفائح طولية (Longitudinal webs) متغيرة العمق (varying depth) خطيا و صفائح لوحية عرضية (Diaphragms) . تم أيجاد مصفوفات الصلادة (Stiffness) لعتبات المشبكات المكافئ ذات الجدران الرقيقة بإدخال تأثير التغير الخطي في العمق ولمختلف تأثيرات تقييد تشويه الالتواء وباستعمال اشتقاق من الطرق التغايرية . هناك طريقتان لإدخال تأثير تقييد الالتواء في مصفوفات الصلادة ،ألاولى: الطريقة الضمنية (Implicit method) (من خلال ثابت الالتواء Torsional constant) .ألثانية : الطريقة الصريحة (Explicit method) كدرجة حرية مستقلة بالنسبة إلى الالتواء تم استخدام طريقتين مختلفتين لاشتقاق مصفوفة صلادة الالتواء (Torsional stiffness matrix) في الأولى كانت بافتراض ألازاحات الحقلية

(Assumed displacement field) و الثانية بالمعادلة التفاضلية للتوازن (Differential equation of equilibrium) . وقد تم الآخذ بنظر الاعتبار تأثير الاجهادات الثانوية للقص (Secondary shearing stresses) كنتيجة تقييد تشويه الالتواء (Warping restraint) . خواص المقطع لعتبات المشبك المكافئ تتغير حسب تغير العمق و هذه الخواص سوف تشتق من خواص المنشأ اللوحي الخلوي الاصلي.بعض الامثلة والتطبيقات قدمت لاثبات كفاءة و دقة الطرق المقترحة، باستخدام برنامج خاص في الحاسوب لتنفيذ التحليلات ، وقد تمت مقارنة النتائج مع تلك المستحصلة من استخدام عناصر محددة (Finite elements) متكونة من صفائح قشرية رقيقة مستوية (Flat plate –shell elements) ضمن البرنامج الجاهز ناستران (NASTRAN) . وكان هناك توافق مقبول في النتائج .

KEY WORDS
Cellular plate,Grillage,Thin-walled structures,Warping effect.

INTRODUCTION

Thin-wall cellular plate structures provide high stiffness and strength with relative small weight.
Some applications of thin-wall closed -cell beam-type structures of varying depth, **Fig.(1),** include supporting tapered box girders for elevated rapid transit systems , for floating structures, in shiphulls, in box gates for dry docks, in large span floors and slabs for factories and in aircraft wings.
According to torsional properties, which are decisive in computing the influence of torsion, it is possible to divide cellular members and box girders into two groups. The *first* embraces cellular girders whose cross sections are restrained , in that the cross sections when twisted do not deflect out of plane(due to warping restraint)
[Vlasov, Waldron, Yoo and Acra,Tan and Montague, Al-Dulaimy ,Al-Dusari] . The second group covers cellular members which do not fulfil this condition. They warp freely or warping is small such that warping restraint has no or little effect on section rigidities [Hambly and Pennls ,Evans and Shanmugam, Muhammed, Al- Sherrawi] .

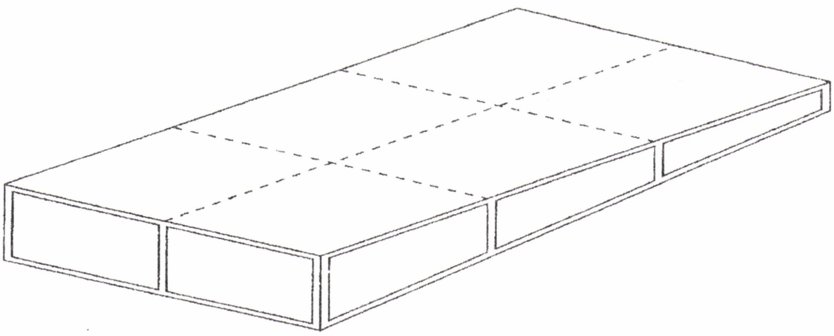


Fig. (1) Thin- wall Cellular Plate structure of Linearly Varying Depth.

GRILLAGE ANALOGY .

A grillage is an assemblage of beam lying in one plane , and having both flexural and torsional rigidities . The grillage analogy is used extensively for the analysis of cellular and ribbed structures.

Grillage Analogy and Co-ordinates System.

Through the consideration of the sign convention and the generalized displacements at each node of the element (1-2) in **Fig. (2-a),** it can be noticed that there are three generalized displacements which are the transverse deflection and two rotations about axes in the plane of the grillage. These degrees of freedom are used when the effect of warping restraint is not included in the stiffness matrix or when it is included implicitly in the torsional stiffness of the grillage members. **Fig. (2-b)** gives the generalized displacements when warping is considered explicitly as another degree of freedom.

The moments and rotations are assumed positive in clockwise direction when viewed from the local coordinates (x,y)(right –hand rule). The forces and displacements are positive downward.

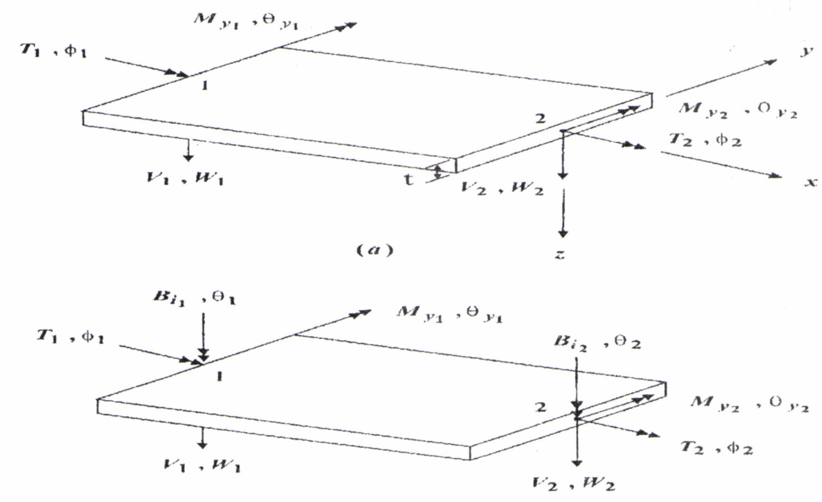


Fig. (2) Discrete element idealization of box(or cellular) beam; (a) warping neglected (or warping implicitly),
(b) warping considered explicitly.

Types of Grillage Mesh.

Two types of grillage mesh are used . In the first , the cellular plate structure is idealized into grillage beams with superimposed flexural and torsional rigidities. The flexural rigidities are obtained from the partitioning webs together with the attached top and bottom cover plates. The torsional and warping rigidities are taken from the warping and torsional rigidities of the adjacent closed cells. The elastic axis of the grillage beams is along the partitioning webs. This is called alternative one.

Additional longitudinal and transverse members (fictitious members) may be added between webs and between diaphragms in the grillage mesh of alternative one in order to provide a better representation of the bending action especially when the webs and diaphragms are widely spaced. These additional members have flanges only and they have flexural and torsional rigidities and no transverse shearing rigidity. This is the second type of mesh.

Stiffness Matrix for Linear Taper Grillage Member (without Warping).

The strain energy for the grillage member (1-2) in **Fig.(2-a)** is:

$$U=\int_0^L \frac{M_y^2}{2EI_y^*}dx+\int_0^L \frac{V^2}{2GA_s^*}dx+\int_0^L \frac{T^2}{2GJ^*}dx \quad (1)$$

where

I_y^*: is the second moment of area (varying along the member).

A_s^* : is the area of the shear-carrying web plate (varying along the member).

J^* : is the torsional constant.

The second moment of area for a grillage member of I-section of linearly varying depth as shown in **Fig. (3)** is derived as follows:

$$\eta = \frac{(h_2 - h_1)}{L}, \quad h(x) = h_1 + \eta . x$$

$$I_y^* = \frac{b_w . h^3(x)}{12} + \frac{2}{(1-\nu^2)}\left[\frac{b_f . t_f^3}{12} + b_f . t_f \left(\frac{h(x)}{2} + \frac{t_f}{2}\right)^2\right] \qquad (2)$$

where the confining effect of Poisson's ratio (υ) for flanges is considered. The expression for I_y^* can be simplified by writing:

$$B = \frac{b_w h_1^3}{12} + \frac{1}{(1-\nu^2)}\left[\frac{2}{3} b_f t_f^3 + \frac{1}{2} b_f t_f h_1^2 + b_f t_f^2 h_1\right]$$

$$C = \left[\frac{b_w}{4} h_1^2 + \frac{1}{(1-\nu^2)}(b_f t_f h_1 + b_f t_f^2)\right].\eta$$

$$D = \left[\frac{b_w}{4} h_1 + \frac{1}{2(1-\nu^2)} b_f t_f\right].\eta^2$$

$$F = \frac{b_w}{12}\eta^3$$

Thus:

$$I_y^*(x) = B + C.x + D.x^2 + F.x^3 \qquad (3)$$

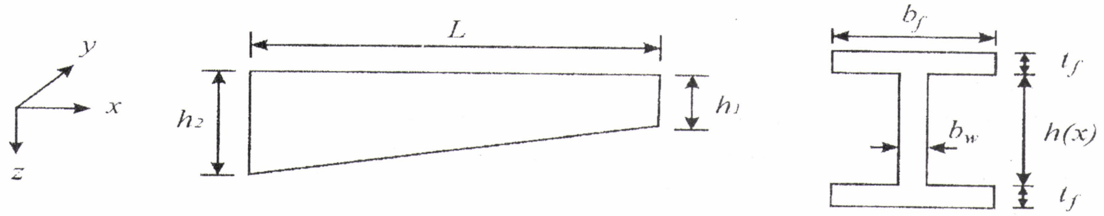


Fig. (3) I-cross section of a taper grillage member.

The stiffness matrix for a taper grillage member (without warping effect) is given below:

$$\begin{bmatrix} V_1 \\ My_1 \\ T_1 \\ V_2 \\ My_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} a & b & 0 & -a & c & 0 \\ & d & 0 & -b & e & 0 \\ & & f & 0 & 0 & -f \\ & & & a & -c & 0 \\ & & & & n & 0 \\ & & & & & f \end{bmatrix} \qquad (4)$$

In the above:

$$a = \frac{1}{BB} \; ; \quad b = \frac{(ZZ_1/ZZ_3)}{BB} \; ; c = \frac{(L - (ZZ_1/ZZ_3))}{BB} \; ; \quad d = \frac{(ZZ_1/ZZ_3)^2}{BB} + \frac{1}{ZZ_3}$$

$$e = \frac{\left(\frac{ZZ_1}{ZZ_3}\right)\left(L - \left(\frac{ZZ_1}{ZZ_3}\right)\right)}{BB} - \frac{1}{ZZ_3}; \qquad f = \frac{1}{ZZ_5} \qquad n = \frac{\left(L - \left(\frac{ZZ_1}{ZZ_3}\right)\right)^2}{BB} + \frac{1}{ZZ_3}$$

where

$$ZZ_1 = \int_o^L \frac{x}{E I_y^*} dx \; ; \quad ZZ_2 = \int_o^L \frac{x^2}{E I_y^*} dx \; ; \; ZZ_3 = \int_o^L \frac{dx}{E I_y^*} \; ; \; ZZ_4 = \int_o^L \frac{dx}{G A_s^*} \; ; \; ZZ_5 = \frac{G J}{L}$$

$$BB = ZZ_2 + ZZ_4 - \frac{ZZ_1^2}{ZZ_3}$$

Stiffness Matrix for a Prismatic Beam Element without Warping Effects.

By using the total strain energy due to bending, shear and torsion for a grillage member , the total stiffness matrix is (Al –Sherrawi):

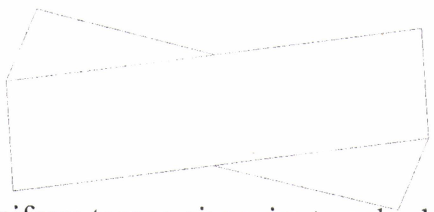
$$\begin{bmatrix} V_1 \\ My_1 \\ T_1 \\ V_2 \\ My_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} a & b & 0 & -a & b & 0 \\ & e & 0 & -b & d & 0 \\ & & e & 0 & 0 & -e \\ & & & a & -b & 0 \\ & & & & c & 0 \\ & & & & & e \end{bmatrix} \begin{bmatrix} W_1 \\ \theta y_1 \\ \phi_1 \\ W_2 \\ \theta y_2 \\ \phi_2 \end{bmatrix} \qquad (5)$$

where

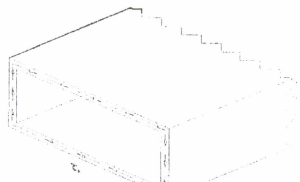
$$a = \frac{12 E I_y}{L^3 (1+\beta)}, \; b = \frac{6 E I_y}{L^2 (1+\beta)}, \; c = \frac{(4+\beta) E I_y}{L(1+\beta)}, \; d = \frac{(2-\beta) E I_y}{L(1+\beta)}, \; e = \frac{G J}{L}, \; \beta = \frac{12 E I_y}{G A_s L^2}$$

Effect of Warping Restraint in The Analysis of Cellular Plates

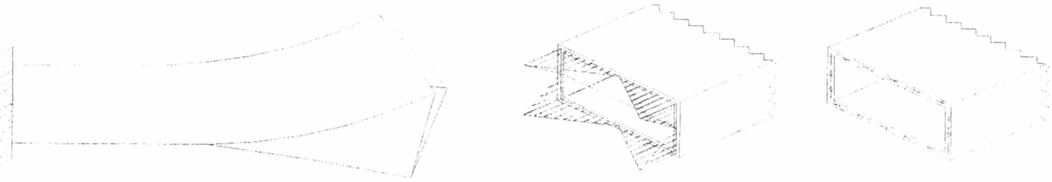
In thin-wall sections, out of plane warping displacements resulting from torsional loading may be generally considerable. However, if warping is in any way restrained, **Fig. (3.1-b),** then a system of self-equilibrium stresses are set up around the cross-section as a result of bending of the individual webs and flanges forming the section. In addition to the physical restraint of warping imposed at the fixed support, full or partial restraint may also be provided by changes in the level of torque along the beam. Thus any application of non-uniform torsion will generally introduce warping restraint stresses in addition to those due to St.- Venant torsion. The resultant of warping restraint stresses so created, is generally referred to as a bimoment. This differs from the more familiar stress resultants, such as axial force and bending moment, in that it cannot be determined from equilibrium conditions alone.



b- Uniform torque gives rise to only shear stresses. Warping stresses are zero and every cross section warps by the same amount. This is Saint Venant torsion.



Saint -Venant Shear stresses on left-hand end.



c- Preventation of warping leads to warping normal stresses bimoment).

Fig. (4) Examples of Saint-Venant and Warping Torsion

Torsional Stiffness with Including Implicitly the Effect of Warping Restraint.

The effect of warping restraint is included implicitly through the use of effective torsion constant (J_{eff}) which will be derived for a thin-walled beam element for different boundary conditions and will be used in the general equation of torsional rotation (ϕ).

The fundamental Equation for the warping displacement (Vlasov)is:

$$\theta'' - k^2 .\theta = -k^2 .\frac{M_t}{G.J}$$

The general solution of this equation is:

$$\theta' = C_1 \sinh kx + C_2 \cosh kx + \frac{M_t}{G.J} \qquad (6)$$

where C_1 and C_2 are unknown constants to be determined from the boundary conditions. After applying the boundary condition, the effective torsional constant when the member has one end restrained is

$$J_{eff} = \frac{J}{(1 - \frac{\mu_\omega}{k\,L} \tanh kL)} \qquad (7)$$

when the member has two ends restrained, the effective torsional constant is given as :

$$\omega_1 = [1 - \frac{\mu_\omega .2(CH - 1)}{k\,L\ .\ SH}] \qquad (8)$$

$$J_{eff} = \frac{J}{\omega_1}$$

where

$SH = \sinh kL$ and $CH = \cosh kL$

$$k=\sqrt{\frac{\mu_{\omega}G.J}{E_{1}.I_{\omega}}}\ ,\quad \mu_{\omega}=1-\frac{J}{I_{c}}\ ,\quad E=\frac{E}{1-\upsilon^{2}}$$

Stiffness Matrix for Thin-Walled Prismatic Beam Element Including Explicitly

Effect of warping restraint.

The member stiffness matrix (8 × 8) can be obtained by assembling the (4 × 4) [K_B] matrix which relates shear and bending and the (4 × 4) [K_T] matrix which relates torsion and warping for a beam element .So it is concluded that the bending and transverse shear could be separated from the torsion and warping effects which are interacted with each other . Thus:

$$[K]=\begin{bmatrix} K_{B(1,} & K_{B(1,2)} & 0 & 0 & K_{B(1,3)} & K_{B(1,4)} & 0 & 0 \\ & K_{B(2,2)} & 0 & 0 & K_{B(2,3)} & K_{B(2,4)} & 0 & 0 \\ & & K_{T(1,1)} & K_{T(1,2)} & 0 & 0 & K_{T(1,3)} & K_{T(1,4)} \\ & & & K_{T(2,2)} & 0 & 0 & K_{T(2,3)} & K_{T(2,4)} \\ & & & & K_{B(3,3)} & K_{B(3,4)} & 0 & 0 \\ & & & & & K_{B(4,4)} & 0 & 0 \\ & & & & & & K_{T(3,3)} & K_{T(3,4)} \\ & & & & & & & K_{T(4,4)} \end{bmatrix} \tag{9}$$

TORSIONAL STIFFNESS MATRIX FOR THIN-WALLED BEAM ELEMENT.

The are two methods for obtaining the torsional stiffness matrix [K_T]:

Method A:

Finite Element Approach (assumed displacement field).

By considering the thin-wall beam element(1-2) shown in **Fig. (2-b),** the nodal displacement and nodal force matrices (related to torsion) are:

$$\{\phi\}=\begin{Bmatrix}\phi_1\\ \theta_1\\ \phi_2\\ \theta_2\end{Bmatrix}_{1-2} \quad \text{and} \quad \{F_{\phi}\}=\begin{Bmatrix}T_1\\ B_{i1}\\ T_2\\ B_{i2}\end{Bmatrix}$$

The expression of the strain energy can be written as

$$U=\frac{1}{2}\int_{o}^{L}\left\{G\,J\,\phi'^{2}+E\,I_{\omega}\,\theta'^{2}+G\,(I_{c}-J\,)(\theta-\phi'^{2})\right\}.dx \tag{10}$$

The corresponding variation equation for the total potential is:

$$\int_{o}^{L}\{G\,J\,\phi'\,\delta\,\phi'+E\,I_{\omega}\theta''\,\delta\,\theta'+G(I_{c}-J)(\theta-\phi')\delta\,(\theta-\phi')-m_{t}\delta\,\phi'\}\,dx$$

$$-T(L)\,\delta\,\phi(L)-T(0)\,\delta\,\phi(0)-B_{i}(L)\,\delta\,\phi(L)-B_{i}(0)\,\delta\,\phi(0)=0$$

In addition, the relation between the twisting angle ϕ and the warping function θ under the homogeneous condition, i.e., the condition with no distributed external torque acting on the element, is also required and this can be shown to be (Al-Sherrawi):

$$\theta=\phi'+\lambda.\phi''' \tag{11}$$

The following polynomial is chosen for the twisting angle ($\phi(x)$:

$$\phi(x)=\lfloor N_\phi \rfloor \{\alpha\} \tag{12}$$

where

$$\lfloor N_\phi \rfloor=\lfloor 1, x, x^2, x^3 \rfloor \tag{13}$$

$$\{\alpha\}=\lfloor \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rfloor$$

From Equation. (11),

$$\theta(x)=\lfloor N_\theta \rfloor \{\alpha\} \tag{14}$$

where

$$\lfloor N_\theta \rfloor=\lfloor 0, 1, 2x, 6x, 3x^2 \rfloor \tag{15}$$

Equations (12) and Equation(14) lead to the following interpolation for ϕ,θ and their derivatives:

$$\phi=\lfloor H_\phi \rfloor\{\Phi\},\ \theta=\lfloor H_\theta \rfloor\{\Phi\},\ \phi'=\lfloor H_1 \rfloor\{\Phi\}, \theta'=\lfloor H_2 \rfloor\{\Phi\},\ \theta-\phi'=\lfloor H_3 \rfloor\{\Phi\} \tag{16}$$

where

$$\lfloor H_\phi \rfloor = \lfloor 1-\frac{12\lambda}{gL}x-\frac{3}{g}x^2+\frac{2}{gL}x^3,\ (1-\frac{6\lambda}{g})x-(\frac{1}{2L}+\frac{3L}{2g})x^2+\frac{x^3}{g},$$
$$\frac{12\lambda}{gL}.x+\frac{3}{g}x^2-\frac{2}{gL}x^3,\ \frac{-6\lambda}{g}x+(\frac{1}{2L}-\frac{3L}{2g})x^2+\frac{x^3}{g} \rfloor$$

$$\lfloor H_\theta \rfloor = \left\lfloor \frac{-12\lambda}{gL}-\frac{6}{g}x+\frac{2}{gL}(6\lambda+3x^2),\ 1-\frac{6\lambda}{g}-(\frac{1}{L}+\frac{3L}{g})x+\frac{(6\lambda+3x^2)}{g},\right.$$
$$\left.\frac{12\lambda}{gL}+\frac{6}{g}x-\frac{2}{gL}(6\lambda+3x^2),\ \frac{-6\lambda}{c}x+(\frac{1}{L}-\frac{3L}{c})x+\frac{(6\lambda+3x^2)}{g}\right\rfloor$$

$$\lfloor H_1 \rfloor=\frac{d}{dx}\lfloor H_\phi \rfloor$$

$$\lfloor H_2 \rfloor=\frac{d}{dx}\lfloor H_\theta \rfloor=\frac{d^2}{dx^2}\lfloor H_\phi \rfloor$$

$$\lfloor H_3 \rfloor=\lfloor H_\theta \rfloor-\lfloor H_1 \rfloor=\lambda\frac{d^3}{dx^3}\lfloor H_\phi \rfloor$$

and

$$g=L^2+12\lambda \tag{17}$$

Substituting Equation(16) into Equation (11) yields:

$$\lfloor K_\phi \rfloor\{\Phi\}=\{F_\phi\}+\{F_m\} \tag{18}$$

where $\lfloor K_\phi \rfloor$ is the torsional stiffness matrix, which is:

$$\lfloor K_\phi \rfloor=\lfloor K_{\phi 1} \rfloor+\lfloor K_{\phi 2} \rfloor+\lfloor K_{\phi 3} \rfloor \tag{19}$$

and

$$\left[K_{\phi_1}\right]=G\,J\int_o^L\left[H_1\right]^T\left[H_1\right]dx=G\,J\begin{bmatrix} \frac{6n}{5Lg^2} & & & \\ \frac{3n}{5g^2}-\frac{1}{2} & \frac{3nL}{10g^2}-\frac{L}{6} & & \\ \frac{-6n}{5Lg^2} & \frac{-3n}{5g^2}+\frac{1}{2} & \frac{6n}{5Lg^2} & \\ \frac{3n}{5g^2}-\frac{1}{2} & \frac{3nL}{10g^2}-\frac{L}{3} & \frac{-3n}{5g^2}+\frac{1}{2} & \frac{3nL}{10g^2}-\frac{L}{6} \end{bmatrix} \tag{20}$$

$$\left[K_{\phi_2}\right]=E\,I_w\int_o^L\left[H_2\right]^T\left[H_2\right]dx=EI_w\begin{bmatrix} \frac{12L}{g^2} & & & \\ \frac{6L^2}{g^2} & \frac{3L^2}{g^2}+\frac{1}{L} & & \\ \frac{-12L}{g^2} & \frac{-6L^2}{g^2} & \frac{12L}{g^2} & \\ \frac{6L^2}{g^2} & \frac{3L^3}{g^2}-\frac{1}{L} & \frac{-6L^2}{g^2} & \frac{3L^3}{g^2}+\frac{1}{L} \end{bmatrix} \tag{21}$$

$$\left[K_{\phi_3}\right]=G(I_c-J)\int_o^L\left[H_3\right]^T\left[H_3\right]dx=G(I_c-J)\lambda^2\begin{bmatrix} \frac{144}{g^2L} & & & \\ \frac{72}{g^2} & \frac{36L}{g^2} & & \\ \frac{-144}{g^2L} & \frac{-72}{g^2} & \frac{144}{g^2L} & \\ \frac{72}{g^2} & \frac{36L}{g^2} & \frac{-72}{g^2} & \frac{36L}{g^2} \end{bmatrix} \tag{22}$$

where

$$n=L^4+20\,\lambda\,L^2+120\,\lambda^2 \tag{23}$$

and $\{F_m\}$ is the equivalent nodal force matrix caused by the distributed external torque, which is:

$$\{F_m\}=\int_o^L m\left[H_\phi\right]^T dx \tag{24}$$

It can be pointed out that $[K\phi_1]$ and $[K\phi_2]$ represent the stiffness associated with free torsion and warping respectively, while $[K\phi_3]$ represent the additional stiffness caused by the secondary shear stress. It can be noticed that if ($\lambda=0$), then the effect of the secondary shear on the warping deformation would be neglected.

Method B: Differential Equation of Equilibrium

The differential Equation for a beam element can be assumed to be (Vlasov):

$$\phi=C_1+C_2x+C_3\sinh(kx)+C_4\cosh(kx)$$

or $\phi(x)=[1 \quad x \quad \sinh(kx) \quad \cosh(kx)]\ \{\alpha\}$

where here $\{\alpha\}=\{C_1, C_2, C_3, C_4\}^T$. From equation (11):

$$\theta(x)=\begin{bmatrix}0 & 1 & \dfrac{k}{\mu_\omega}\cosh(kx) & \dfrac{k}{\mu_\omega}\sinh(kx)\end{bmatrix}\{\alpha\}$$

The boundary displacements are given by:

$$[\Phi]=[A]\ \{\alpha\}$$

or in a matrix form:

$$\begin{Bmatrix}\phi_1\\ \theta_1\\ \phi_2\\ \theta_2\end{Bmatrix}=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & \dfrac{k}{\mu_w} & 0\\ 0 & L & S & C\\ 0 & 1 & \dfrac{k}{\mu_w}C & \dfrac{k}{\mu_w}S\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\\ C_3\\ C_4\end{Bmatrix} \tag{26}$$

Then

$$[A]^{-1}=\frac{1}{2-2CH+\dfrac{k}{\mu_w}SH}\begin{bmatrix}1-CH+\dfrac{kL}{\mu_w}SH & \dfrac{kL}{\mu_w}CH-SH & 1-CH & SH-\dfrac{kL}{\mu_w}\\ \dfrac{-k}{\mu_w}SH & -(1-CH) & \dfrac{kL}{\mu_w}SH & -(1-CH)\\ SH & 1-CH+\dfrac{kL}{\mu_w}SH & -SH & CH-1\\ 1-CH & \dfrac{kL}{\mu_w}CH-SH & CH+1 & -(\dfrac{kL}{\mu_w}-SH)\end{bmatrix} \tag{27}$$

The force boundary conditions are defined by:

$$[F_\phi]=[B]\{\alpha\}$$

$$\begin{Bmatrix}T_1\\ B_{t1}\\ T_2\\ B_{t2}\end{Bmatrix}=\begin{bmatrix}EI_w\theta_1''-GJ\phi_1'\\ EI_w\theta_1'\\ -EI_w\theta_2''+GJ\phi_2'\\ -EI_w\theta_2'\end{bmatrix}=GJ\begin{bmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -S & -C\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\\ C_3\\ C_4\end{Bmatrix} \tag{28}$$

Equations (27) and (28) are used together to form the element stiffness matrix as follow:

$$[F_\phi]=[B]\ \{\alpha\}=[B]\ [A]^{-1}\ \{\Phi\}$$

Hence;

$$[K_\phi]=[B]\ [A]^{-1} \tag{29}$$

So, finally:

$$[K_\phi]=\overline{GJ}\begin{bmatrix}k\ SH & \mu_\omega(1-CH) & -k\ SH & \mu_\omega(1-CH)\\ & \dfrac{\mu_\omega}{k}(kLCH-\mu_\omega SH) & -\mu_\omega(1-CH) & \dfrac{\mu_\omega}{k}(\mu_\omega SH-k)\\ & & k\ SH & -\mu_\omega(1-CH)\\ & & & \dfrac{\mu_\omega}{k}(kLCH-\mu_\omega SH)\end{bmatrix} \tag{30}$$

where:

$$\overline{GJ}=\frac{GJ}{[2\,\mu_\omega\,(1-CH)+kL\ SH)]}$$

TRANSFORMATION MATRIX.

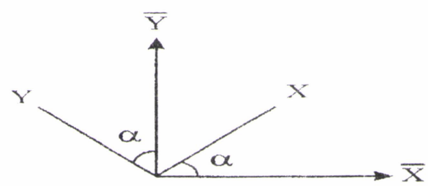
The stiffness matrix must be transformed from the local to the global coordinate system for all members. The transformation matrix for a beam element without warping as an independent degree of freedom is:

$$[T]=\begin{bmatrix}[T'] & 0\\ 0 & [T']\end{bmatrix} \quad \text{where} \quad [T']=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}$$

The angle of inclination between the local and the global axes is (α).

When the effect of warping is included as another degree of freedom, the grillage member will have eight degrees of freedom. So in this case ,the transformation matrix will be :

$$[T]=\begin{bmatrix}[T'] & & & \\ 0 & 1 & & \\ 0 & 0 & [T'] & \\ 0 & 0 & 0 & 1\end{bmatrix}$$



ELASTIC SECTION RIGIDITIES OF GRILLAGE MEMBERS.

These elastic section rigidities are as follows:

1- Bending (or flexural) rigidity ($E I_{(y)}$).

2- Pure torsional rigidity ($G J$)[or modified when warping is included implicitly].

3- Shearing rigidity ($G A_s$).

4- Warping rigidity ($E I_w$)[when warping degree of freedom is considered explicitly].

Bending or Flexural Rigidity.

The moment of inertia of the section is equal to the sum of moments of inertia for the flanges and web (about the centroid of the I-section).

$$I = I_f + I_w$$

The effect of shear lag and Poisson's ratio are to be considered. The shear lag effect is included by using the effective flange width[Moffat and Dowling]. The confining effect of Poisson's ratio is incorporated by multiplying (I_f) by the factor of $\frac{1}{(1-\nu^2)}$ as suggested by [Mohammed].

For the fictitious members, the moments of inertia are determined by the same manner but neglecting the contribution of the web.

Shearing Rigidities (Distortion).

Thin-wall cells distort by transverse shearing forces unless sufficient transverse diaphragms or frames brace them. Although such distortion behavior cannot be reproduced precisely in a flat grillage, an approximation can be obtained by giving the grillage members an effective cross-sectional area which results in shear deflections approximating to those of the Vierendeel.

Evans and Shanmugam , Mohammed and Al-Sherrawi proposed a shear area equal to the full area of the web or the diaphragm to the grillage members.

Torsional Rigidity.

Torsion in a cellular plate structure is resisted by shear flows around the cells in both webs and flanges. The shear flow at an intermediate web represents the difference between the shear flows of adjacent cells. Thus, a set of simultaneous equations in the shear flows (q) can be obtained. The rate of twist $(d\phi / dx)$ can be written as:

$$\frac{d\phi}{dx} = \frac{1}{G.\Omega}\left[q\oint\frac{ds}{t}\right] \qquad (31)$$

where Ω = double area enclosed by the middle line of the section,

$(\Omega = \oint r_i .ds)$ and (t)is the thickness of wall and (s) is measured along the wall For

(i th) cell :

$$\frac{d\phi_i}{dx} = \frac{1}{G.\Omega_i}\left[q_i \oint\frac{ds}{t} - \sum_{K=1}^{m} q_K \int_K \frac{ds}{t}\right] \qquad (32)$$

where the last term refers to the parts of the adjacent cells(in the boundary of the (ith) cell). In order to make a simplification [7], Equation (32) can be expressed as :

$$\Omega_i = \left[\psi_i \int_i \frac{ds}{t} - \sum_{K=1}^{m} \psi_K \int_K \frac{ds}{t}\right] \qquad (33)$$

where (ψ_i) is a geometric quantity and is easily determined for a given cell(i):

$$\psi_i = \frac{q}{G.d\phi / dx} \qquad (34)$$

So, the torsional constant for the cell (J_i) is

$$J_i = \psi_i .\Omega_i \qquad (35)$$

The torsional constant of the cross-section of the substitute grillage member is taken as one-half the torsional constant of the actual cross-section. The other half represents the transverse shear stiffness. The concept of using half value of torsional rigidity is based on the fact that torsion in a cellular plate occurs in two directions while in a grillage member in one direction.
Finally the torsional constant of the whole section is the sum of these (n) constants.

$$J = \sum_{i=1}^{n} J_i \qquad (36)$$

Warping Rigidity.

The sectorial area (ω) at any point on the thin-wall closed section [Waldron ,1986] is given by ;

$$\omega_i = \int_o^s (r_i - \frac{\psi_i}{t})\, ds \qquad (37)$$

where

$$\psi = \frac{\Omega}{\oint \frac{ds}{t}} = \frac{q}{G\phi'} \qquad (38)$$

Then, the geometrical warping constant (I_ω) and the sectorial moment of area (s_ω) are evaluated as following:

$$I_\omega = \oint_A \omega_{(s)}^2 .dA$$

$$S_{\omega(s)} = \int_o^s \omega_{(s)} .dA \qquad (39)$$

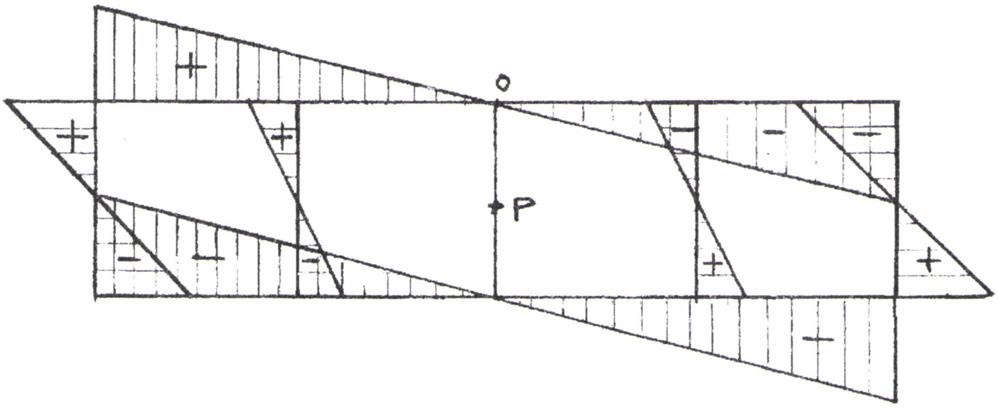


Fig.(5) Determination of the sectorial area (ω)for non-uniform multi-cell section.

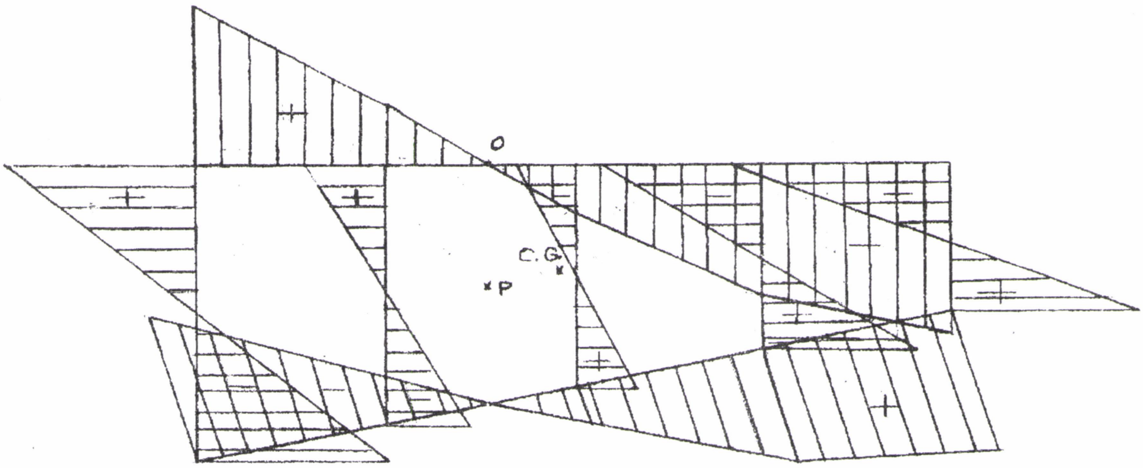


Fig.(6) Determination of the sectorial area((ω)for non-uniform (tapered) multi-cell section.

The sectorial area (ω) can be estimated for the uniform thin – wall cellular box section (having two axes of symmetry) as shown in **Fig.(5).** In non- uniform thin- walled box section (where no axes of symmetry exit), the sectorial area diagram can be expected to be as shown in **Fig.(6).**

A computer program is developed in the present work to serve for this purpose [for the torsional and warping properties (J, ψ, I_ω) of the section in the transverse direction where the position of the shear center is known, but it can be used just for calculating (J and ψ) for the cross-section in the longitudinal direction (varying depth) because the position of shear center must be determined by hand].

INTERPRTATION OF OUTPUT.

The output obtained from the grillage analysis includes the deflection under the applied loading and the system of internal forces that could be utilized in determining the associated stresses. By using the formula of the elementary bending theory of beams, the bending stress (σ_x) at distance (z) from the neutral axis can be calculated from the grillage moment (M_y) as follows:

$$\sigma_x = \frac{M_y . z}{I_y} \tag{40}$$

The additional normal stress due to bimoment (σ_ω) is calculated by :

$$\sigma_\omega = \frac{B_{i_T}\ \omega_{(s)}}{I_{\omega T}} \tag{41}$$

where (B_{i_T}) is the total bimoment given by:

$$B_{i_T} = \sum_{i=1}^{n} B_{i_i} \tag{42}$$

in which (B_{i_i}) is the bimoment for a grillage member (i) which will result from the grillage analysis, $\omega(s)$ is the warping function (or the sectorial area) at the point where the normal stress is to be calculated and ($I_{\omega T}$) is the warping moment of inertia for the whole section.

The transverse shear flow (q_b) (from bending action) can be calculated by using the simple beam theory:

$$q_b = \frac{V . Q}{I_y} \tag{43}$$

The shear flow (q_t) due to warping torsion moment consists of two parts. One part (the primary distribution) has the same transverse distribution as in Saint-Venant torsion (q'). The other part (the secondary distribution) is in equilibrium with the warping normal stresses (q'').

$$q_t = q' + q'' \tag{44}$$

where the primary shear flow is:

$$q' = \frac{T}{\Omega} \tag{45}$$

and the secondary shear flow is:

$$q'' = \frac{-B_i'}{I_\omega} \overline{S}_\omega \tag{46}$$

where B_i` is the first derivative of the bimoment (B_i= $-E_1 . I_\omega . d\theta/dx$) and

$$\overline{S_\omega} = S_\omega - \frac{\oint S_\omega . dA}{\Omega}$$

The total shear flow is the summation of the shear flows calculated as above. The total shearing stresses in each plate component represents the total shear flow divided by the thickness of the plate.

APPLICATION

A square steel cellular plate structure simply supported at four edges is analyzed by the grillage method . The same example is analyzed by using the three –dimensional finite plate-shell elements of the Structural Analysis Program (MSC/ NASTRAN).

Two grillage alternatives of meshes are considered for the grillage method, the first type include the main members and the other with fictitious members in one direction .Nine points of loads of 2000Kn each are applied on the interior intersection points as shown in **Fig.(7)** . Large loads are used mainly for amplifying the deflections and stresses.

Table (1) shows the comparison of the maximum deflections at center (point B) and the maximum normal stresses at the same position (point B). By the comparison of the results obtained from the

grillage methods with those gained from the finite elements, it can be concluded that the differences in the maximum deflections
decreased from (24.04 %) to (5.67 and 7.69 %) when shifting from the first type of grillage mesh to the second and third types (which include the effects of warping restraint).
When fictitious members are used, it can be noticed that the differences in the maximum deflections decreased from (16.538 %) to (4.8 % and 3.846 %).

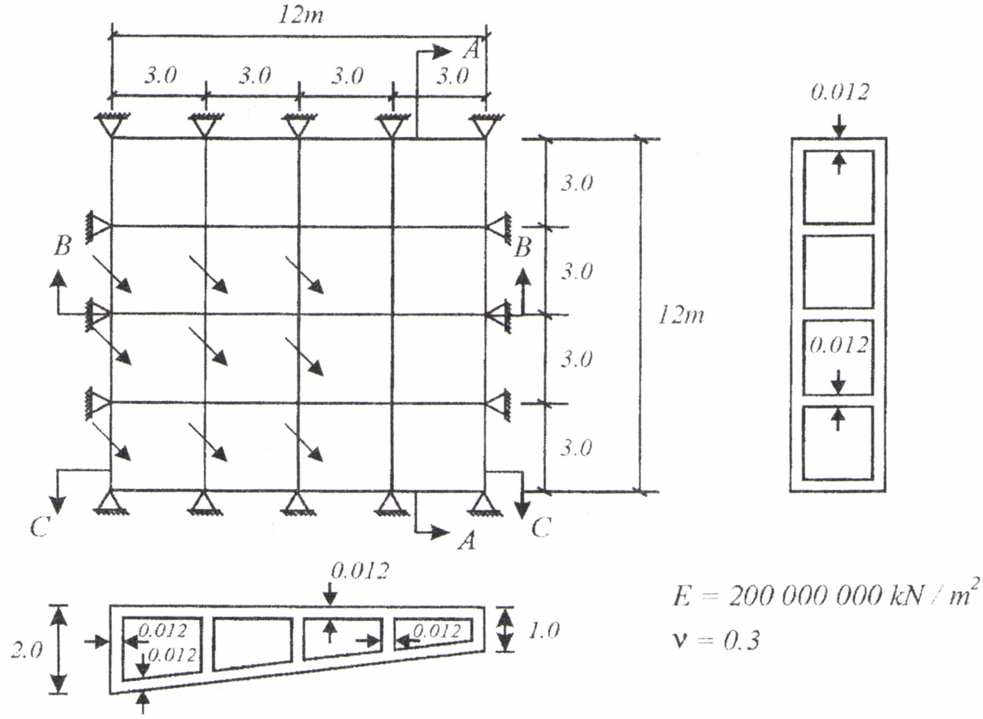
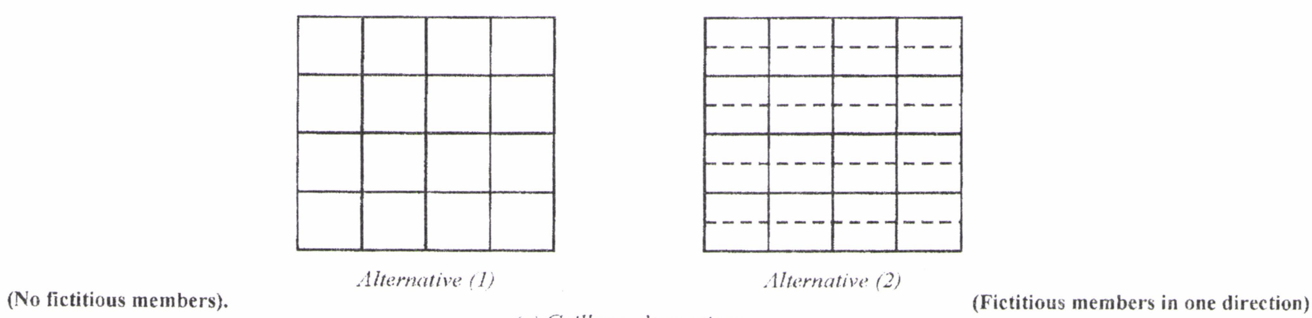
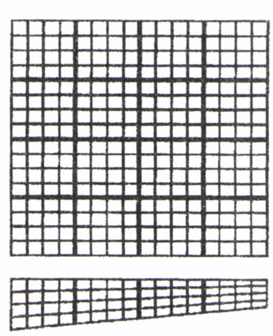


Fig. (7) Steel cellular plate structure simply supported at four edges.



(a) Grillage alternatives.



(b) Finite element mesh.

Fig. (8) Grillage representation and mesh size in finite element method

Table (1) Comparison of maximum vertical deflections and normal stresses at section (A-A) for the cellular plate structure simply supported at four edges.

Methods of analysis		*Mesh size*	*Max. deflections (mm)*	*Max. normal stresses (MPa)*
Grillage methods	*Without warping (3 D.O.F.)*	*Alt 1*	*12.9*	*112.1*
		Alt 2	*12.12*	*92*
	With implicit warp (3 D.O.F.)	*Alt 1*	*10.99*	*102*
		Alt 2	*10.6*	*96.35*
	With explicit warp (4 D.O.F.)	*Alt 1*	*11.2*	*107*
		Alt 2	*10.8*	*99.4*
Finite element method			*10.4*	*106.784*

The maximum normal stress gained from the grillage method without warping (first type) is (112.1 MPa) and that obtained from the other types are (102 and 107 MPa) and from the finite element (106.784 MPa).So the differences are respectively (4.978 %, 4.48 % and 0.202 %) and when using the fictitious members, the differences increased because the value became under- estimated properly due to distribution of stresses over wider areas.

CONCLUSIONS.

1- Effect of warping restraint need be included to improve the results of deflections, torsion's and normal stresses because this effect decreases the percentage of differences and the improvement increases when using fictitious members. This effect is noticeable when the cellular plate structure is under eccentric loads.(producing higher torsion).

2- Inclusion of the effect of restraint of warping either implicitly through the effective torsional constant or explicitly as another degree of freedom makes the grillage analogy more efficient. It can be observed that the difference in results from the implicit and explicit considerations is not more than 2 %. The values of deflections and normal stresses, which are obtained from the implicit method, are almost, less than those from the explicit method. As well , the resulting bimoments cannot be estimated if the restrained warping is considered implicitly, but it is recommended to use the implicit method in order to minimize the total degrees of freedom.

3- The effect of restraint against warping at the ends of a beam produces a significant increase in the torsional stiffness especially when the beam becomes short.

4- Effect of secondary shearing stresses due to warping restraint is considered in addition to the shearing stresses in the sections due to bending and due to applied torque (Saint- Venant shear stress).Some improvement is gained.

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NOTATIONS

As Shear area of grillage member and equal to the full area of the web.

b_f Width of flange plate.

b_w Thickness of web plate.

d The central distance between top and bottom flanges.

E,G Modulus of elasticity and rigidity.

$\{F_m\}, \{F_\phi\}$ Equivalent nodal force matrix.

h(x) Varying depth of webs along the length .

h_1, h_2 Depths of the web at ends.

I_C Moment of area about the centroid

I_ω Principal sectorial moment of inertia of the section or warping constant .

I_y	Second moment of area due to bending.
J, J_{eff}	Torsional and effective torsional constant .
L	Length of tapered grillage member.
K	Stiffness coefficient.
k	Shear area correction factor.
M_t	Torsional moment.
M_y	Bending moment about local y-axes.
m_t	Distributed torque.
q	Shear flow or the force per unit length .
q_t	Total shear flow.
$q^{`}, q^{``}$	Primary and secondary shear flow.
r_t	Perpendicular distance from the shear center to the tangent at the section under consideration
S	Curvilinear coordinate on the middle line of the section.
$S_{\omega}(s)$	Principal sectorial static moment .
T	Torque at any section.
[T]	Transformation matrix.
T_f	Thickness of flange plate.
u(s)	Warping displacement .
U	Strain energy.
V	Shear force
W	Vertical displacement X,Y,Z Cartesian coordinate axes.

Greek symbols

Ω	Double area enclosed by the middle line of the section.
θ	Warping function.
θ_y	Bending rotation about local y- axes.
μ_{ω}	Warping shear parameter.
$\xi_{(s)}$	Angular displacement.
σ	Normal stress.
σ_{ω}	Normal stress due to warping
ϕ	Angle of twist
$\phi^{`}, \delta\phi^{`}$	Real and virtual rate of twist.
ψ	Torsional function
ω	Sectorial area .