THREE DIMENSIONAL INTERSECTIONS BY ANGULAR OBSERVATIONS

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ABSTRACT
The research introduces a new mathematical model for computing the position of points in space (X, Y, Z) by the intersection of inclined angles derived from the measured horizontal and vertical angles using spherical trigonometry principles with the aid of the conventional surveying instrument (Theodolite), The most probable value (m.p.v) of the position of a point should be computed by the method of least squares when the redundancy in observations exists and then it provides us with the precision indices for the computed coordinates.

الخلاصة
يقدم البحث المطروح نموذجا رياضيا جديدا لحساب مواقع النقاط في الفضاء الثلاثي الأبعاد بواسطة تقاطع الزوايا المائلة المشتقة من الزوايا الأفقية والرأسية باستخدام مبادئ المثلثات الكروية وباستخدام الجهاز التقليدي لقياس الزوايا (الثيودولايت).هذا وتحسب القيمة الأكثر احتمالية لموقع النقطة من خلال طريقة اقل المربعات عندما يكون الفائض في القياس موجودا وبالتالي ستكون مقاييس الدقة للإحداثيات المحسوبة متوفرة أيضا.

KEY WORDS
Intersection – vector dot product-inclined angle-spherical trigonometry-least square

INTRODUCTION
The determination of the position of a point in space is a common problem in surveying engineering, which is done either by linear measurements (distances) or by angular measurements (angles) or both. The first type of measurements is accomplished by using electronic distance measuring instruments (EDM), and the second type is accomplished by the theodolite for observing vertical and horizontal angles. Sometimes it is too hard or impossible to occupy the unknown point by an EDM reflector, therefore angular observations will be more applicable and effective.
This research introduces a new method to determine the position of a point in three dimensional space system (x, y, z) by angular observations which is achieved by measuring horizontal and vertical angles to the unknown point from the fixed points, thence the developed mathematical model offers the simultaneous determination of the three coordinates of the unknown point instead of determining the planimetric coordinates (x, y) separated from the (z) coordinate of the point.
It should be noted that the new method utilizes the same concept of Church method in Photogrammetry which is used to compute the exterior orientation parameters of photos. The research deals with the problem considering redundant angular observations which must exceed three derived inclined angles.
Many researchers tried to solve the problem in several procedures. However, the famous solution

was introduced by (Ball) 1973, this researcher take into account the measured horizontal and vertical angles directly but his procedure has the disadvantage of producing the unknown distances between the fixed points and the unknown point besides the three unknown coordinates (x, y, z), but he overcome this problem by matrices maneuver but the solution is still tedious and complicated. The derived method is too active, promising, and offers a wide range of applications in surveying engineering that deals with positions in space (x, y, z) system such as monitoring the deformation of large structures, when connected to local or referenced datum.

THE PROPOSED MATHEMATICAL MODEL

In vectors geometry it is a well-known principles that the dot product of any two vectors is written as follows:

$$A.B = /A/ \; /B/ \cos(\varphi) \qquad (1)$$

Where
$A = Xai + Yaj + Zak$
$B = Xbi + Ybj + Zbk$
φ = the angle between vectors A and B.
/A/, /B/ = the length of vectors A and B respectively.

It follows that for any angle (φ) in space, the following relationship could be used for the evaluation of φ

$$\cos\phi = \frac{X_A X_B + Y_A Y_B + Z_A Z_B}{/A//B/} \qquad (2)$$

Now, in space intersection problem **Fig. (1)** it is obvious that if angle (φ) is measured in plane 1P2 then the position of the unknown point (p) could be determined in three dimensions.

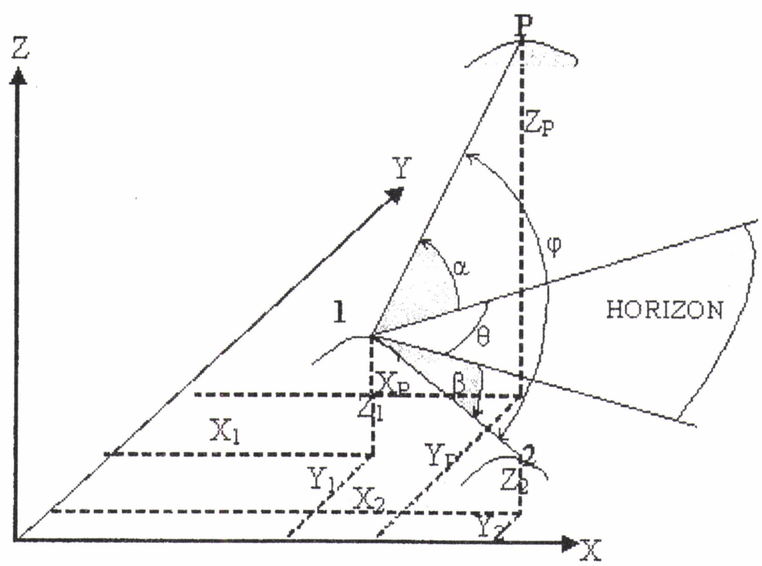

Fig. (1)

Angle (φ) as noticed is an inclined angle, thus it could be evaluated by measuring horizontal angle (θ) and vertical angles (α, β) then applying the principles of spherical trigonometry as discussed in appendix A.

$\cos\varphi = \cos\theta \cos\alpha \cos\beta + \sin\alpha \sin\beta$ (3)

According to **Fig. (1)**, the following equations are obtained:-

$$\cos(\phi 1) = \frac{(x_1 - x_{p^o})\Delta x_{12} + (y_1 - y_{p^o})\Delta y_{12} + (z_1 - z_{p^o})\Delta z_{12}}{l_{12} l_{1p^o}}$$

$$\cos(\phi 2) = \frac{(x_2 - x_{p^o})\Delta x_{23} + (y_2 - y_{p^o})\Delta y_{23} + (z_2 - z_{p^o})\Delta z_{23}}{l_{23} l_{2p^o}} \quad (4)$$

$$\cos(\phi_n) = \frac{(x_n - x_{p^o})\Delta x_{n1} + (y_n - y_{p^o})\Delta y_{n1} + (z_n - z_{p^o})\Delta z_{n1}}{l_{n1} l_{np^o}}$$

Where point's 1,2,3…n are fixed points, While x_{p^o}, y_{p^o} and z_{p^o} represent the approximate position of P, which could be obtained by solving any plane triangle to determine (x_{p^o}, y_{p^o}) since the horizontal angles are measured then determining the z_{p^o} by the measured vertical angles.

LINEARIZATION AND LEAST SQUARE METHOD

To solve the problem, the system of non linear equations eq. (4) must be linearized by using Tylor's theorem, and it is obvious that three observations are the required number of observations to have a unique solution. It must be noted that three control points are enough to solve the problem if more than two inclined angles (φ) are measured, but when a redundant observations are available with (n) total observations, then the problem must be solved by the method of least square to find the most probable value of the position of point P.

The linearized form of the equations is developed by partially differentiating the function with respect to the parameters neglecting higher order terms.

The final linearized equations are:

$F1 = b_{11}\, dx_p + b_{12}\, dy_p + b_{13}\, dz_p$

$F2 = b_{21}\, dx_p + b_{22}\, dy_p + b_{23}\, dz_p$

. . (5)

. .

. .

$Fn = b_{n1}\, dx_p + b_{n2}\, dy_p + b_{n3}\, dz_p$

In eq. (5), n represents the number of observations and the b's (coefficients) and F's (constant terms) are defined as follows (zero subscript signifies that the estimated values for x_p, y_p, z_p are used in the computations): -

$b_{11} = dx_{21} - A_1 (x_{p^o} - x_1)$

$b_{12} = dy_{21} - A_1 (y_{p^o} - y_1)$

$b_{13} = dz_{21} - A_1 (z_{p^o} - z_1)$

Where

$A_1 = [\cos(\varphi_1)(l_{12} / l_{p^o 1})]$

$$l_{p^o 1} = (dx_{p^o 1}^{\ 2} + dy_{p^o 1}^{\ 2} + dz_{p^o 1}^{\ 2})^{\frac{1}{2}}$$

(6)

$$b_{21} = dx_{32} - A_2 (x_{p^o} - x_2)$$

$$b_{22} = dy_{32} - A_2 (y_{p^o} - y_2)$$

$$b_{23} = dz_{32} - A_2 (z_{p^o} - z_2)$$

Where

$$A_2 = [\cos(\varphi_2)(l_{23} / l_{p^o 2})]$$

$$l_{2p^o} = (dx_{p^o 1}^{\ 2} + dy_{p^o 2}^{\ 2} + dz_{p^o 2}^{\ 2})^{\frac{1}{2}}$$

$$b_{n1} = dx_{1n} - A_n (x_{p^o} - x_n)$$

$$b_{n2} = dy_{1n} - A_n (y_{p^o} - y_n)$$

$$b_{n3} = dz_{1n} - A_n (z_{p^o} - z_n)$$

Where

$$A_n = [\cos(\varphi_n)(l_{1n} / l_{np^o})]$$

$$l_{np^o} = (dx_{p^o n}^{\ 2} + dy_{p^o n}^{\ 2} + dz_{p^o n}^{\ 2})^{\frac{1}{2}}$$

Also

$$f1 = [l_{12} l_{1p^o} \cos\varphi_1 - dx_{12} (x_1 - x_{p^o}) - dy_{12} (y_1 - y_{p^o}) - dz_{12} (z_1 - z_{p^o})]$$

$$f2 = [l_{23} l_{2p^o} \cos\varphi_2 - dx_{23} (x_2 - x_{p^o}) - dy_{23} (y_2 - y_{p^o}) - dz_{23} (z_2 - z_{p^o})] \qquad (7)$$

.

.

$$f_n = [l_{n1} l_{np^o} \cos\varphi_n - dx_{n1} (x_n - x_{p^o}) - dy_{n1} (y_n - y_{p^o}) - dz_{n1} (z_n - z_{p^o})]$$

Eq.(4) are solved by the method of least square for adjustment by observation equations to obtain the corrections dx_p,dy_p and dz_p to the approximate values x_{p^o} , y_{p^o} and z_{p^o} .

The solution in matrix form is:

$$X = (B^T W B)^{-1} (B^T W F) \qquad (8)$$

Where

X = (3*1) column matrix of corrections.

B = (n*3) matrix of partial derivatives of the function with respect to the unknowns.

W = (n*n) square matrix of weights.

F = (n*1) column matrix of constant terms.

The corrections are added to x_{p^o} , y_{p^o} and z_{p^o} to obtain the improved estimates and the procedure is repeated to determine new corrections to be added to the improved estimates. The iterations are continued until the corrections become negligible. The iterations of the solutions decreased when the initial estimates are chosen properly.

ILLUSTRATIVE EXAMPLE

An experimental test was adopted to evaluate the new developed method by using real data taken from observations made previously to survey the dimensions and verticality of the Baghdad university building.

As shown in the **Fig. (2)**, four control points (1,2,3, and 4) were used and an angular intersection is achieved from these fixed points to the unknown point (p) lying at one of the top corners of the building. These derived angles were φ_1, φ_2, φ_3, and φ_4 from the observed horizontal and vertical angles (αi , βi, θi) by the cosine rule (see appendix A) as illustrated in **Table (1)**.

Table (1) Measured Angles

No. of points	X meter	Y meter	Z meter H.I meter	Measured horizontal angle From~to	Measured vertical angle ~ point P	Computed Inclined Angle φ
1	100.000	100.000	30.000 1.540	2~p 18°05'02''	52°31'59''	55°02'29''
2	162.291	118.148	29.948 1.535	4~p 154°59'23''	72°18'07''	106°8'32''
3	309.158	88.882	31.120 1.630	1~p 10°50'12''	21°39'14''	24°40'44''
4	312.411	99.932	31.682 1.385	3~p 83°37'36''	21°37'59''	85°34'45''

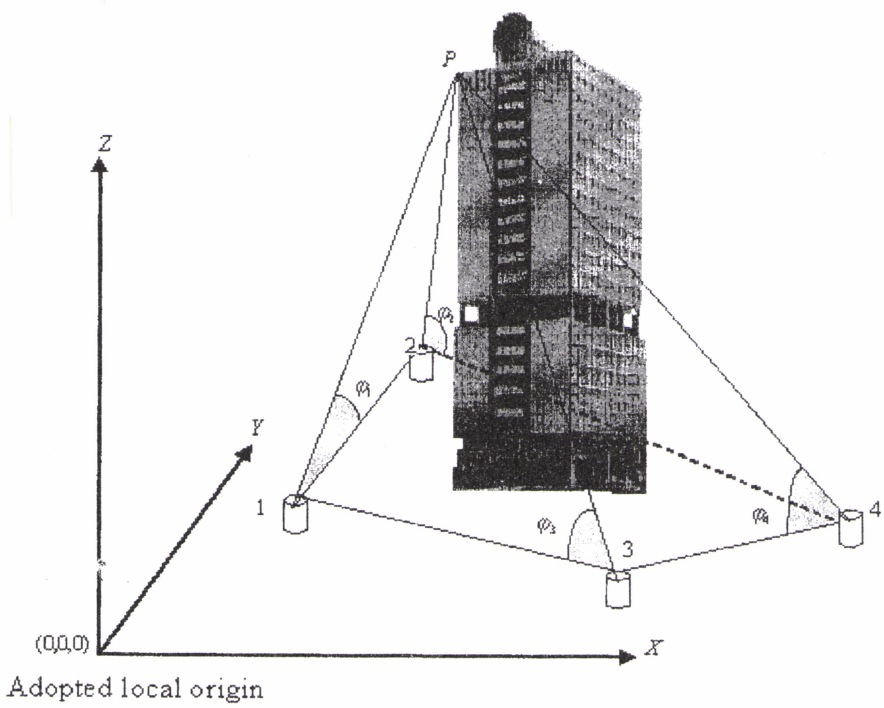

Fig. (2)

It must be noted that the adopted coordinate system was local (x,y,h) in this test, and as stated before the type of the coordinate system is not essential since the derived model is applicable to any three dimensional model.

The approximate position of P is selected randomly as x_{p^o} =110.000, y_{p^o} =110.000, and z_{p^o} = 100.000 in meters.

The solution is illustrated in **Table (2)**, which represents the five iterations needed to solve the problem until the amount of corrections became insignificant.

An appropriate program was prepared by using the MATLAB programming language on a personal computer to solve the problem and to compute the final adjusted coordinates (x_p, y_p, and z_p) as illustrated in appendix B.

Table (2) Improved Computed Positions for Five Iterations

No. of iterations	ΔX_m	ΔY_m	ΔZ_m	X_p	Y_p	Z_p
1	32.597	18.742	6.758	142.597	128.742	106.760
2	0.127	4.502	-6.899	142.724	133.244	99.860
3	0.028	-0.205	-0.367	142.752	133.038	99.493
4	0.000	-0.001	-0.001	142.752	133.037	99.492
5	0.000	0.000	0.000	142.752	133.037	99.492

CONCLUSIONS

It is obvious from the experimental test that the developed model is too efficient in determining the three dimensional position of points because of the fast convergence and the accuracy indices that could be provided in the solution with a minimum required number of control points (three points) through the method of least squares which means that this model could be adopted for the applications of dimensional surveying and monitoring the deformation of large structures like dams, bridges and towers, also the method could be adopted in photogrammetric applications.

REFERENCES

Ball W.E. (1973), Three Dimensional Intersections-a Vector Approach, journal of surveying and mapping division, ASCE, vol.99, No.SVI, pp. 63-78

Paul r.wolf, (1983), Elements of Photogrammetry, McGraw-Hill, USA.2nd edition, pp.240-245.

NOMENCLATURE

A B: vectors in space.

X_A, Y_A, Z_A: position of point A in space.

X_B, Y_B, and Z_B: position of point B in space.

x_p, y_p, and z_p: approximate position of the unknown point P.

li: spatial distance.

b's: coefficients.
α, β: vertical angles.
θ : Horizontal angle.
φ : Inclined angle between A&B

APPENDIX A

Computation of Inclined Angles

Let the line OA represents the gradient of a plane inclined to the horizontal OA' whilst line OB is the gradient of a second plane inclined at B' to the horizontal OB' **Fig. (A1).** The horizontal angle between the lines OA and OB, i.e. $A'OB'$, is θ whilst the inclined lines themselves lie in a common plane and subtend an angle φ in the plane AOB. Assuming the point above O is Z (the zenith); the arcs $A'AZ$, $B'BZ$ and AB are all part of great circles. In the spherical triangle AZB, Z=θ, b=90°-α, a=90°-β and z =φ. By the cosine rule.

$$\cos\theta = \frac{\cos\phi - \sin\alpha\sin\beta}{\cos\alpha\cos\beta} \qquad \text{A1}$$

$$\cos\theta = \frac{\cos\phi - \cos(90-\alpha)\cos(90-\beta)}{\sin(90-\alpha)\sin(90-\beta)} \qquad \text{A2}$$

And thus

$$\cos\varphi = \cos\theta\cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \text{A3}$$

This means that the inclined angle φ could be computed by using measured vertical and horizontal angles (α, β, θ)

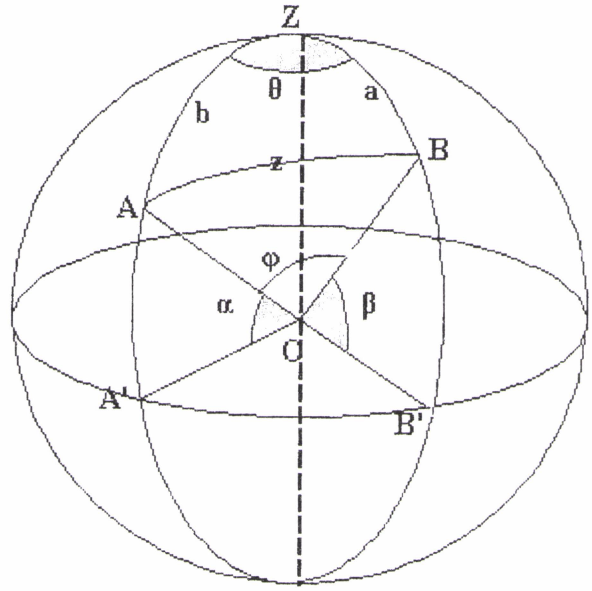

Fig. (A1)

APPENDIX B

Computer Program

```
%%%%%%%%%%%%% 3 DIMENSIONAL INTERSECTION%%%%%%%%%%%%
xp=110;yp=110;zp=100;n=4;
for i=1:4
   i
x(i,1)=input('enter x coordinates');
y(i,1)=input ('enter y coordinates=');
z(i,1)=input('enter z coordinates=');
end
q=[55.2593;24.7991;106.2923;85.1425];
q=q.*pi/180;r='chose';c=0;wz=0;
while c~=4
   n=4;
dx(1,1)=(x(2,1)-x(1,1)); dx(2,1)=(x(1,1)-x(3,1));dx(3,1)=(x(4,1)-x(2,1)); dx(4,1)=(x(3,1)-x(4,1));
dy(1,1)=(y(2,1)-y(1,1));dy(2,1)=(y(1,1)-y(3,1));dy(3,1)=(y(4,1)-y(2,1));dy(4,1)=(y(3,1)-y(4,1));
dz(1,1)=(z(2,1)-z(1,1));dz(2,1)=(z(1,1)-z(3,1));dz(3,1)=(z(4,1)-z(2,1));dz(4,1)=(z(3,1)-z(4,1));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
dxp(1,1)=xp-x(1,1);dxp(2,1)=xp-x(3,1);dxp(3,1)=xp-x(2,1);dxp(4,1)=xp-x(4,1);
dyp(1,1)=yp-y(1,1);dyp(2,1)=yp-y(3,1);dyp(3,1)=yp-y(2,1);dyp(4,1)=yp-y(4,1);
dzp(1,1)=zp-z(1,1);dzp(2,1)=zp-z(3,1);dzp(3,1)=zp-z(2,1);dzp(4,1)=zp-z(4,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for i=1:4
lp(i,1)=sqrt(dxp(i,1)^2+dyp(i,1)^2+dzp(i,1)^2);
end
for i=1:n
l(i,1)=sqrt(dx(i,1)^2+dy(i,1)^2+dz(i,1)^2);
a(i,1)=(cos(q(i,1))*(l(i,1)/lp(i,1)));
end
for i=1:n
b1(i,1)=(dx(i,1)-a(i,1)*dxp(i,1));
b2(i,1)=(dy(i,1)-a(i,1)*dyp(i,1));
b3(i,1)=(dz(i,1)-a(i,1)*dzp(i,1));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 f(i,1)=(l(i,1)*lp(i,1)*cos(q(i,1))-dx(i,1)*dxp(i,1)-dy(i,1)*dyp(i,1)-dz(i,1)*dzp(i,1));
end
b=[b1 b2 b3];
btb=b'*b;
btf=b'*f;
x1=inv(btb)*btf
if x1<=.00001
c=4;
end
format long g
xp=xp+x1(1,1)
yp=yp+x1(2,1)
zp=zp+x1(3,1)
wz=wz+1
end
```

```
v=b*x1-f;
vv=v'*v;
s=vv*inv(btb)
for i=1:3
vcov(i,1)=sqrt(s(i,i));
end
```