A New Analytic Method to Tune a Fractional Order PID Controller

Omar Waleed Abdulwahhab
Assistant Professor
College of Engineering-University of Baghdad
Email: omarwaleed110@gmail.com

Dr. Nizar Hadi Abbas
Assistant Professor
College of Engineering-University of Baghdad
Email: drnizaralmsaodi@gmail.com

ABSTRACT

This paper proposes a new method to tune a fractional order PID controller. This method utilizes both the analytic and numeric approach to determine the controller parameters. The control design specifications that must be achieved by the control system are gain crossover frequency, phase margin, and peak magnitude at the resonant frequency, where the latter is a new design specification suggested by this paper. These specifications results in three equations in five unknown variables. Assuming that certain relations exist between two variables and discretizing one of them, a performance index can be evaluated and the optimal controller parameters that minimize this performance index are selected. As a case study, a third order linear time invariant system is taken as a process to be controlled and the proposed method is applied to design the controller. The resultant control system exactly fulfills the control design specification, a feature that is laked in numerical design methods. Through matlab simulation, the step response of the closed loop system with the proposed controller and a conventional PID controller demonstrate the performance of the system in terms of time domain transient response specifications (rise time, overshoot, and settling time).

Keywords: Fractional order PID controller; gain crossover frequency; phase margin; peak magnitude at resonant frequency.

طريقة تحليلية جديدة لتنغيم المسيطر التناسبي-التكاملي -التفاضلي ذو الرتبة الكسرية

عمر وليد عبدالوهاب
أستاذ مساعد
كلية الهندسة – جامعة بغداد

د. نزار هادي عباس
أستاذ مساعد
كلية الهندسة – جامعة بغداد

الخلاصة

يقترح هذا البحث طريقة جديدة لتنغيم المسيطر التناسبي-التكاملي -التفاضلي ذو الرتبة الكسرية. هذه الطريقة تستثمر الطريقة التحليلية والطريقة العددية لايجاد معاملات المسيطر. خصائص النظام المصمم تشمل تردد تقاطع الربح وهامش الطور والقيمة العظمى للمقدار عند التردد الرنيني (وهذه هي الخاصية الجديدة المقترحة في هذا البحث). هذه الخصائص تعطي ثلاث معادلات في خمسة مجاهيل. بافتراض وجود علاقة بين متغيرين وتقطيع احدهما يمكن ايجاد مؤشر الاداء وايجاد معاملات المسيطر الامثل. كمثال تم أخذ نظام من الرتبة الثالثة خطي ذو معاملات ثابتة وتم تطبيق الطريقة المقترحة عليه فكان النظام الناتج يطابق تماما الخصائص المطلوبة وهو ما تفتقد اليه الطرق العددية لتصميم المسيطر. من خلال المحاكاة تم اختبار استجابة النظام لادخال خطوة للمسيطر المقترح واخر تقليدي ذو رتبة صحيحة وتم الاستدلال على أداء النظام بدلالة مواصفات الاستجابة الانتقالية في مجال الزمن.

1. INTRODUCTION

The fractional order PID controller (also called $PI^{\lambda}D^{\mu}$) was proposed by **Podlubny, 1994** and **Podliubny, 1999** and has a transfer function

$$C(s) = K_p + \frac{K_I}{s^{\lambda}} + K_D s^{\mu} \qquad (1)$$

where K_p, K_I, $K_D \in R$ and λ and $\mu \in R^+$ are the parameters of the controller that must be tuned. Parameters λ and μ increase the degree of freedom in tuning the controller, which makes the design of the control system more flexible. Fractional order PID controllers have less sensitivity to parameter variation due to these two additional parameters, **Zhao, et al., 2005**. Since the $PI^{\lambda}D^{\mu}$ controller has five parameters, up to five design specifications can be fulfilled by this controller, while the PID controller can fulfill up to three design specifications.

In general, there are two approaches to tune the $PI^{\lambda}D^{\mu}$ controller, analytical and numerical **Valerio, and Costa, 2010**. In **Zhao, et. al, 2005** and **Caponetto, et al., 2004**, the controller parameters were derived analytically to achieve gain (phase) margin and phase (gain) crossover frequency specifications. In **Monje, et al., 2008**, one of the equations was taken as an objective function and the rest of the specifications were taken as constraints. In **Bhisikar, et. Al, 2014, Wang, et al, 2015,** and **Badri, 2015**, the set of equations were solved numerically. **Badri, and Tavazoei, 2013** solved the equations graphically by finding the intersection point of two curves. **Sadati, 2007** used a performance index in time domain to determine the controller parameters, while **Tepljakov, et al., 2015** used a performance index in frequency domain to determine these parameters. **Lazarevic, 2013** tuned the controller parameters using genetic algoritm to minimize a performance index.The objective functions in such problems have complex surfaces such that the analytic methods of optimization often fail **Dorcak, et al, 2006**. Another approach to tune a fractional order PID controller was given in **Xue, et al., 2006, Khalil, et. Al., 2009**, and **Joshi and Talange, 2013**, by taking cerain values for the fractional order of integration and differentiation and finding the optimal values for the remaining gain parameters.

The remaining of this paper is organized as follows: In section 2, the proposed tuning method is presented, in section 3, a design and simulation example is presented to demonstrate the application of this method, and in section 4, the conclusions are drawn from the simulation results.

2. THE PROPOSE TUNING METHOD

Consider the unity feedback control system shown in **Fig. 1**. The transfer functions of the controller and the plant are $C(s)$ and $P(s)$, respectively. The sinusoidal transfer function of the controller is

$$C(j\omega) = K_p + \frac{K_I}{(j\omega)^{\lambda}} + K_D(j\omega)^{\mu} = K_p + K_I(j\omega)^{-\lambda} + K_D(j\omega)^{\mu} = K_p + K_I j^{-\lambda}\omega^{-\lambda} + K_D j^{\mu}\omega^{\mu}$$

$$= K_p + K_I(\cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right))^{-\lambda}\omega^{-\lambda} + K_D(\cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right))^{\mu}\omega^{\mu}$$

$$C(j\omega) = K_p + K_I(\cos\left(\frac{\lambda\pi}{2}\right) + j\sin\left(\frac{\lambda\pi}{2}\right))\omega^{-\lambda} + K_D(\cos\left(\frac{\mu\pi}{2}\right) + j\sin\left(\frac{\mu\pi}{2}\right))\omega^{\mu}$$

$$C(j\omega) = K_P + K_I \cos\frac{\lambda\pi}{2}\omega^{-\lambda} + K_D \cos\frac{\mu\pi}{2}\omega^{\mu} + j(-K_I \sin\frac{\lambda\pi}{2}\omega^{-\lambda} + K_D \sin\frac{\mu\pi}{2}\omega^{\mu}) \quad (2)$$

$$C(j\omega_c)P(j\omega_c) = 1\angle(\phi_m - \pi) = 1e^{j(\phi_m-\pi)} = e^{j\phi_m}e^{j(-\pi)} = e^{j\phi_m}(\cos(-\pi) + j\sin(-\pi))$$

$$C(j\omega_c)P(j\omega_c) = e^{j\phi_m}\big(-1 + j(0)\big) = -e^{-j\phi_m}$$

$$C(j\omega_c) = \frac{-e^{-j\phi_m}}{P(j\omega_c)} \quad (3)$$

Substituting Eq. (2) in Eq. (3) and equating the real part and imaginary part of both sides yields

$$K_P + K_I \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \cos\frac{\mu\pi}{2}\omega_c^{\mu} = \Re(\frac{-e^{-j\phi_m}}{P(j\omega_c)}) \quad (4)$$

$$-K_I \sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \sin\frac{\mu\pi}{2}\omega_c^{\mu} = \Im(\frac{-e^{-j\phi_m}}{P(j\omega_c)}) \quad (5)$$

The steady state error should be zero; therefore, $\lambda > 0$ (Final value theorem). The two relations $\mu = 1 - \lambda$ and $\mu = \lambda$ are assumed to exist between μ and λ; the reason for choosing these relations is to get the sine and cosine functions for the same angle, namely, $\frac{\lambda\pi}{2}$.

i) First, assume that

$$\mu = 1 - \lambda \quad (6)$$

There is one degree of freedom in choosing λ and μ (choose one and evaluate the other) as shown in **Fig. 2**. Substituting Eq. (6) in Eq. (4) and Eq. (5) yields

$$K_P + K_I \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \cos\frac{(1-\lambda)\pi}{2}\omega_c^{1-\lambda} = \Re(\frac{-e^{-j\phi_m}}{P(j\omega_c)})$$

$$K_P + K_I \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \sin\frac{\lambda\pi}{2}\omega_c\omega^{-\lambda} = \Re(\frac{-e^{-j\phi_m}}{P(j\omega_c)}) \quad (7)$$

$$-K_I \sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \sin\frac{(1-\lambda)\pi}{2}\omega_c^{1-\lambda} = \Im(\frac{-e^{-j\phi_m}}{P(j\omega_c)})$$

$$-K_I \sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \cos\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} = \Im(\frac{-e^{-j\phi_m}}{P(j\omega_c)}) \quad (8)$$

Now, take discrete values of λ, from 0 to 1, say 0.01, 0.02, …, 1 (100 values). For each value of λ, solve Eq. (7) and Eq. (8) for K_I and K_D in terms of K_P and λ.

$$K_I = \frac{\begin{vmatrix} \Re\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) - K_P & \sin\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} \\ \Im\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) & \cos\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} \end{vmatrix}}{\begin{vmatrix} \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \sin\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} \\ -\sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \cos\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} \end{vmatrix}} = \frac{\begin{vmatrix} \Re\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) - K_P & \sin\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} \\ \Im\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) & \cos\frac{\lambda\pi}{2}\omega_c\omega_c^{-\lambda} \end{vmatrix}}{\omega_c\omega_c^{-2\lambda}} \quad (9)$$

$$K_D = \frac{\begin{vmatrix} \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \Re\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) - K_P \\ -\sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \Im\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) \end{vmatrix}}{\omega_c\omega_c^{-2\lambda}} \quad (10)$$

At the resonant frequency of the plant ω_r, the magnitude of the open loop transfer function is

$|C(j\omega_r)P(j\omega_r)| = M_r \Rightarrow |C(j\omega_r)| = \frac{M_r}{|P(j\omega_r)|}$. Let $\frac{M_r}{|P(j\omega_r)|} = k$

$$\sqrt{(K_P + K_I\cos\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D\sin\frac{\lambda\pi}{2}\omega_r\omega^{-\lambda})^2 + (-K_I\sin\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D\cos\frac{\lambda\pi}{2}\omega_r\omega^{-\lambda})^2} = k_1 \quad (11)$$

Squaring both sides of Eq. (11) yields

$$(K_P + K_I\cos\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D\sin\frac{\lambda\pi}{2}\omega_r\omega^{-\lambda})^2 + (-K_I\sin\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D\cos\frac{\lambda\pi}{2}\omega_r\omega^{-\lambda})^2 = k_1^2 \quad (12)$$

Substituting Eq. (9) and Eq. (10) in Eq. (12), the value of K_P is obtained as follows

$$K_P = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \quad (13)$$

where

$$a_1 = 1 - 2\alpha^{\lambda}\cos^2\frac{\lambda\pi}{2} - 2\alpha^{\lambda-1}\sin^2\frac{\lambda\pi}{2} + \alpha^{2\lambda}\cos^2\frac{\lambda\pi}{2} + \alpha^{2\lambda-2}\sin^2\frac{\lambda\pi}{2}$$

$$\begin{aligned} b_1 = {} & 2\alpha^{\lambda}R\cos^2\frac{\lambda\pi}{2} - 2\alpha^{\lambda}I\sin\frac{\lambda\pi}{2}\cos\frac{\lambda\pi}{2} + 2\alpha^{\lambda-1}I\sin\frac{\lambda\pi}{2}\cos\frac{\lambda\pi}{2} + 2\alpha^{\lambda-1}R\sin^2\frac{\lambda\pi}{2} \\ & - 2\alpha^{2\lambda}R\cos^2\frac{\lambda\pi}{2} + 2\alpha^{2\lambda}I\sin\frac{\lambda\pi}{2}\cos\frac{\lambda\pi}{2} - 2\alpha^{2\lambda-2}I\sin\frac{\lambda\pi}{2}\cos\frac{\lambda\pi}{2} \\ & - 2\alpha^{2\lambda-2}R\sin^2\frac{\lambda\pi}{2} \end{aligned}$$

$$\begin{aligned} c_1 = {} & \alpha^{2\lambda}R^2\cos^2\frac{\lambda\pi}{2} - 2\alpha^{2\lambda}RI\sin\frac{\lambda\pi}{2}\cos\frac{\lambda\pi}{2} + \alpha^{2\lambda}I^2\sin^2\frac{\lambda\pi}{2} + \alpha^{2\lambda-2}I^2\cos^2\frac{\lambda\pi}{2} \\ & + 2\alpha^{2\lambda-2}RI\sin\frac{\lambda\pi}{2}\cos\frac{\lambda\pi}{2} + \alpha^{2\lambda-2}R^2\sin^2\frac{\lambda\pi}{2} - k_1^2 \end{aligned}$$

$\alpha = \frac{\omega_c}{\omega_r}, \; R = \Re(\frac{-e^{-j\phi m}}{P(j\omega_c)}), \; I = \Im(\frac{-e^{-j\phi m}}{P(j\omega_c)})$

The performance index that will be used to select the optimal value of λ

$$P_1(\lambda) = \int_0^\infty e^2(t,\lambda)dt \tag{14}$$

ii) Second, assume that

$$\mu = \lambda \tag{15}$$

Substituting Eq. (15) in Eq. (4) and Eq. (5) yields

$$K_P + K_I \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \cos\frac{\lambda\pi}{2}\omega_c^{\lambda} = \Re(\frac{-e^{-j\phi m}}{P(j\omega_c)}) \tag{16}$$

$$-K_I \sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} + K_D \sin\frac{\lambda\pi}{2}\omega_c^{\lambda} = \Im(\frac{-e^{-j\phi m}}{P(j\omega_c)}) \tag{17}$$

For the same discrete values of λ, solve Eq. (16) and Eq. (17) for K_I and K_D in terms of K_P and λ.

$$K_I = \frac{\begin{vmatrix} \Re\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) - K_P & \cos\frac{\lambda\pi}{2}\omega_c^{\lambda} \\ \Im\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) & \sin\frac{\lambda\pi}{2}\omega_c^{\lambda} \end{vmatrix}}{\begin{vmatrix} \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \cos\frac{\lambda\pi}{2}\omega_c^{\lambda} \\ -\sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \sin\frac{\lambda\pi}{2}\omega_c^{\lambda} \end{vmatrix}} = \frac{\begin{vmatrix} \Re\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) - K_P & \cos\frac{\lambda\pi}{2}\omega_c^{\lambda} \\ \Im\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) & \sin\frac{\lambda\pi}{2}\omega_c^{\lambda} \end{vmatrix}}{\sin\lambda\pi} \tag{18}$$

$$K_D = \frac{\begin{vmatrix} \cos\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \Re\left(\frac{-e^{-j\phi m}}{P(j\omega_c)}\right) - K_P \\ -\sin\frac{\lambda\pi}{2}\omega_c^{-\lambda} & \Im(\frac{-e^{-j\phi m}}{P(j\omega_c)}) \end{vmatrix}}{\sin\lambda\pi} \tag{19}$$

$$|C(j\omega_r)| = \frac{M_r}{|P(j\omega_r)|} = k$$

$$\sqrt{(K_P + K_I \cos\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D \cos\frac{\lambda\pi}{2}\omega_r^{\lambda})^2 + (-K_I \sin\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D \sin\frac{\lambda\pi}{2}\omega_r^{\lambda})^2} = k_2 \tag{20}$$

Squaring both sides of Eq. (20) yields

$$(K_P + K_I \cos\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D \cos\frac{\lambda\pi}{2}\omega_r^{\lambda})^2 + (-K_I \sin\frac{\lambda\pi}{2}\omega_r^{-\lambda} + K_D \sin\frac{\lambda\pi}{2}\omega_r^{\lambda})^2 = k_2^2 \tag{21}$$

Substituting Eq. (18) and Eq. (19) in Eq. (21), the value of K_P is obtained as follows

$$K_P = \frac{-b_2 \pm \sqrt{b_2^2 - 4a_2 c_2}}{2a_2} \tag{22}$$

where

$$a_2 = 1 + \frac{\alpha^{2\lambda}}{4\cos^2\frac{\lambda\pi}{2}} + \frac{1}{4\alpha^{2\lambda}\cos^2\frac{\lambda\pi}{2}} - \alpha^{\lambda} - \frac{1}{\alpha^{\lambda}} + \frac{\cos\lambda\pi}{2\cos^2\frac{\lambda\pi}{2}}$$

$$\begin{aligned} b_2 = \frac{\alpha^{2\lambda}}{\sin^2\lambda\pi}\left(-2R\sin^2\frac{\lambda\pi}{2} + I\sin\lambda\pi\right) + \frac{1}{\alpha^{2\lambda}\sin^2\lambda\pi}\left(-I\sin\lambda\pi - 2R\sin^2\frac{\lambda\pi}{2}\right) \\ + \frac{\alpha^{\lambda}\cos\frac{\lambda\pi}{2}}{\sin\lambda\pi}\left(R\sin\frac{\lambda\pi}{2} - I\cos\frac{\lambda\pi}{2}\right) + \frac{2\cos\frac{\lambda\pi}{2}}{\alpha^{\lambda}\sin\lambda\pi}\left(I\cos\frac{\lambda\pi}{2} + R\sin\frac{\lambda\pi}{2}\right) \\ + \frac{2\cos\lambda\pi}{\sin^2\lambda\pi}\left(-2R\sin^2\frac{\lambda\pi}{2}\right) \end{aligned}$$

$$\begin{aligned} c_2 = \frac{\alpha^{2\lambda}}{\sin^2\lambda\pi}\left(R^2\sin^2\frac{\lambda\pi}{2} - RI\sin\lambda\pi + I^2\cos^2\frac{\lambda\pi}{2}\right) \\ + \frac{1}{\alpha^{2\lambda}\sin^2\lambda\pi}\left(I^2\cos^2\frac{\lambda\pi}{2} + RI\sin\lambda\pi + R^2\sin^2\frac{\lambda\pi}{2}\right) \\ + \frac{2\cos\lambda\pi}{\sin^2\lambda\pi}\left(R^2\sin^2\frac{\lambda\pi}{2} - I^2\cos^2\frac{\lambda\pi}{2}\right) - k_2^2 \end{aligned}$$

The performance index in this case is

$$P_2(\lambda) = \int_0^{\infty} e^2(t,\lambda)dt \tag{23}$$

The optimal value of λ is

$$\lambda^* = \arg\{\min[\min_{\lambda}(P_1(\lambda)), \min_{\lambda}(P_2(\lambda))]\}$$

and the optimal value of μ is

$$\mu^* = \begin{cases} 1-\lambda^* & \text{if } \min_{\lambda}(P_1(\lambda)) \le \min_{\lambda}(P_2(\lambda)) \\ \lambda^* & \text{if } \min_{\lambda}(P_1(\lambda)) > \min_{\lambda}(P_2(\lambda)) \end{cases}$$

Thus, the transfer function of the controller is

$$C(s) = K_p + \frac{K_I}{s^{\lambda^*}} + K_D s^{\mu^*} \tag{24}$$

3. DESIGN AND SIMULATION

Consider applying the proposed design procedure to the plant given by the transfer function

$$P(s) = \frac{1}{s^3 + 0.6675s^2 + 2.8985s + 0.561}$$

The resultant $PI^{\lambda}D^{\mu}$ controller is

$$C(s) = -0.2374 + \frac{0.5484}{s^{0.615}} + 0.2317s^{0.615} \quad (25)$$

which resuts when ($\mu = \lambda$). The bode plot of the open loop transfer function is shown in **Fig. 4**. The control design specifications ($\phi_m = 60^{\circ}$, $\omega_c = 0.3$, and $M_r = 0.1$) are all achieved by this controller.

The step response of the close loop system with this $PI^{\lambda^*}D^{\mu^*}$ controller is shown in **Fig. 5**. For the purpose of comparison, a conventional PID controller is designed using matlab *pidtune* command, which designes a PID controller for a given transfer function. The transfer function of this controller is

$$C(s) = 0.167 + \frac{0.127}{s} \quad (26)$$

The step response of the close loop system with the PID controller is shown in **Fig. 6**. **Table** 1 shows the transient response specifications of the two systems. The system with the $PI^{\lambda}D^{\mu}$ controller has better percentage overshoot, delay time, and rise time than that with the PID controller, while the settling time is much greater. This is because the specifications that are fulfilled by the $PI^{\lambda}D^{\mu}$ controller are the gain crossover frequency which enhances the rise time and delay time, and the phase margin which enhances the percentage overshoot.

4. CONCLUSIONS

A conclusion can be drawn from the design procedure of this paper that if the domain of some of the design variables is restricted by a certain mathematical relation (restriction) between these variables (λ and μ in this case), an analytic, rather than a numerical, solution can be obtained. Unlike the optimization problems that may be subjected to certain constraints (such as the parameters should be positive), the analytic solution gives exact solution of the design specification equations; this solution may be positive, negative, or even a complex number. This is evident in the value of K_P, which is negative. While a $PI^{\lambda}D^{\mu}$ controller that is tuned by optimization techniques can fulfill the design specifications with some sufficiently small error, the analytically tuned $PI^{\lambda}D^{\mu}$ controller fulfill the design specifications exactly, since it is the solution of a set of simultaneous equations; this is evident in enhancing the percentage overshoot and rise time of the closed loop system.

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NOMENCLATURE

e= error between the desired and actual output.
j= imaginary unit (= $\sqrt{-1}$).
K_D= derivative gain of the PID or $PI^{\lambda}D^{\mu}$ controller, dimensionless.
K_I= integral gain of the PID or $PI^{\lambda}D^{\mu}$ controller, dimensionless.
K_p= proportional gain of the PID or $PI^{\lambda}D^{\mu}$ controller, dimensionless.
M_r=magnitude of the open loop transfer function at the resonant frequency, dimensionless.
$P_1(.)$, $P_2(.)$= performance indices.
R= set of real numbers.
R^+= set of positive real numbers.
λ= fractional order of integration of the $PI^{\lambda}D^{\mu}$ controller, dimensionless.
λ^*= optimal value of λ, dimensionless.
μ= fractional order of differentiation of the $PI^{\lambda}D^{\mu}$ controller dimensionless.
μ^*= optimal value of μ, dimensionless.
ω_c= gain crossover frequency of the open loop transfer function, radian/s.
ω_r= resonance frequency of the plant, radian/s.
$\Re$= real part of a complex number.
$\Im$= imaginary part of a complex number.

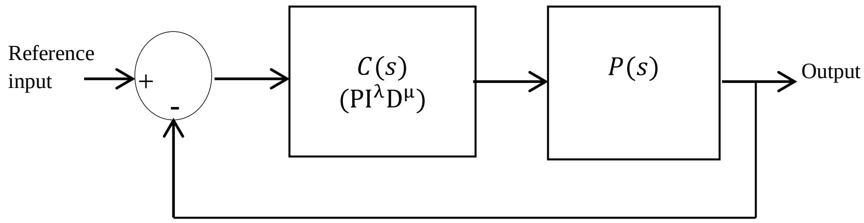

Figure 1. Unity feedback control system.

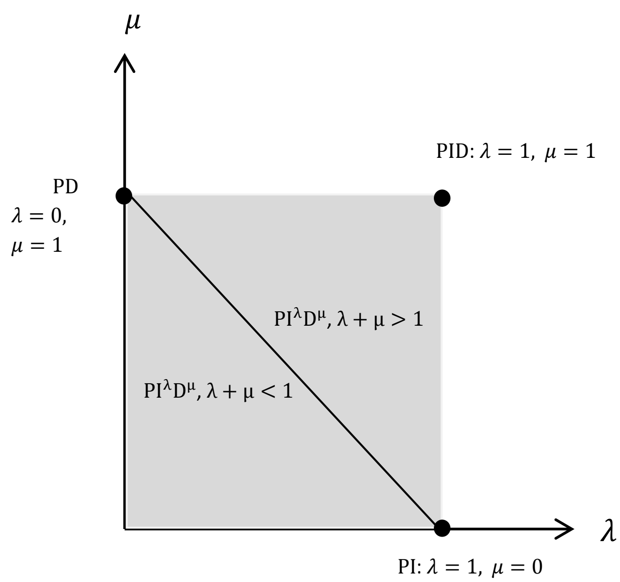

Figure 2. Relation between λ and μ: $\lambda + \mu = 1$.

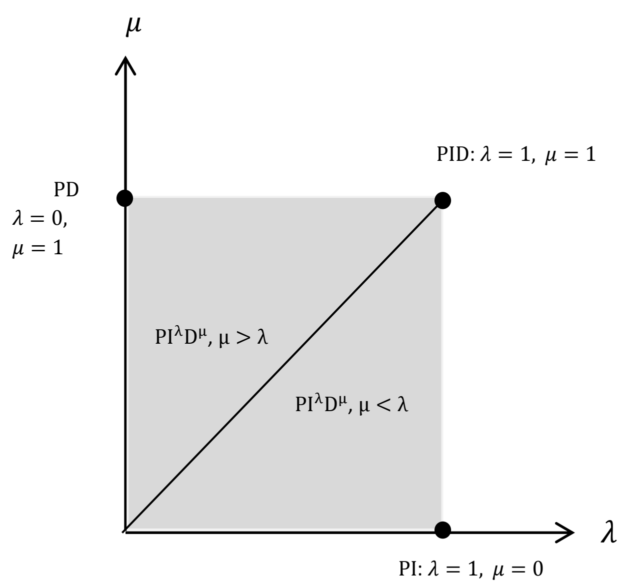

Figure 3. Relation between λ and μ: $\mu = \lambda$.

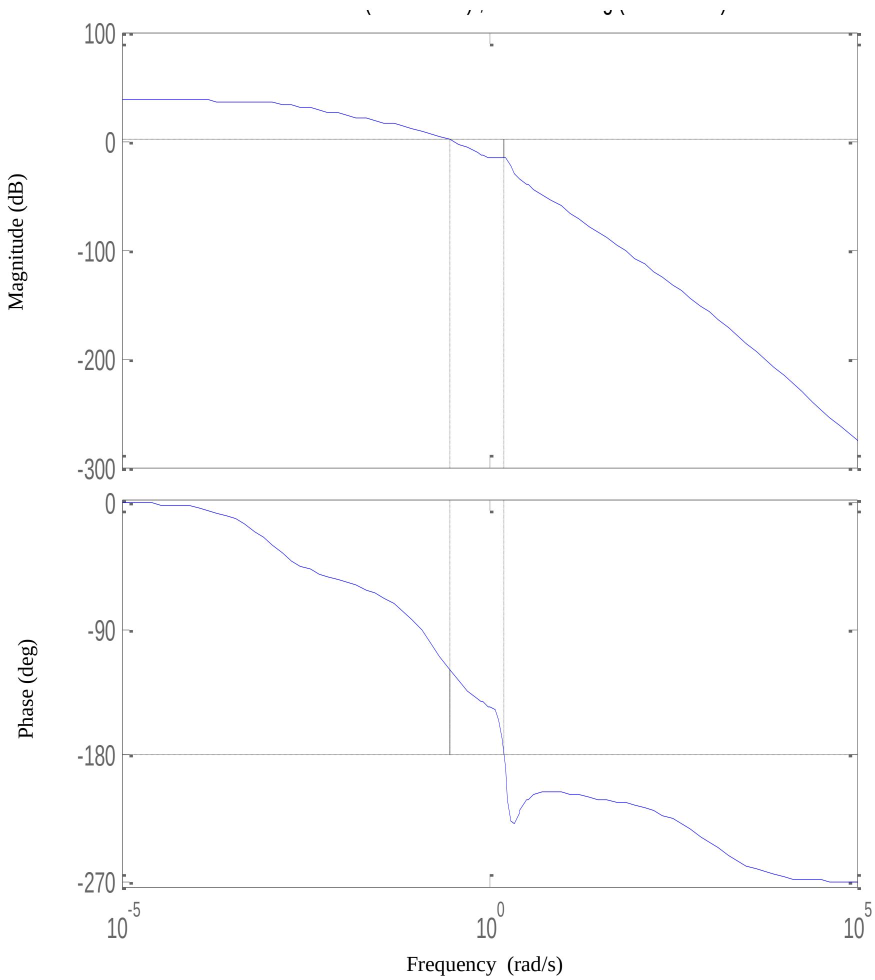

Figure 4. Bode plot of the open loop transfer function with $PI^{\lambda}D^{\mu}$ controller.

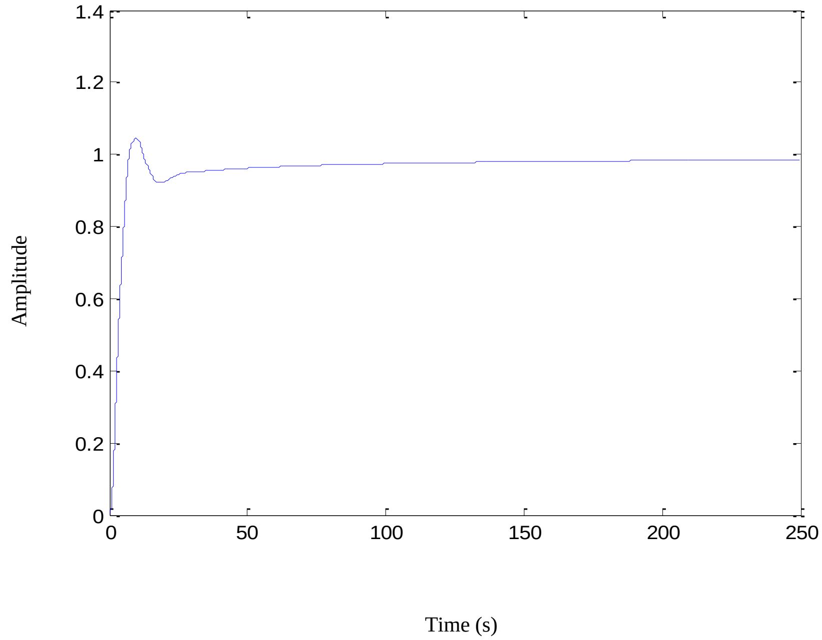

Figure 5. Step response of the close loop system with PI$^{\lambda^*}$D$^{\mu^*}$ controller.

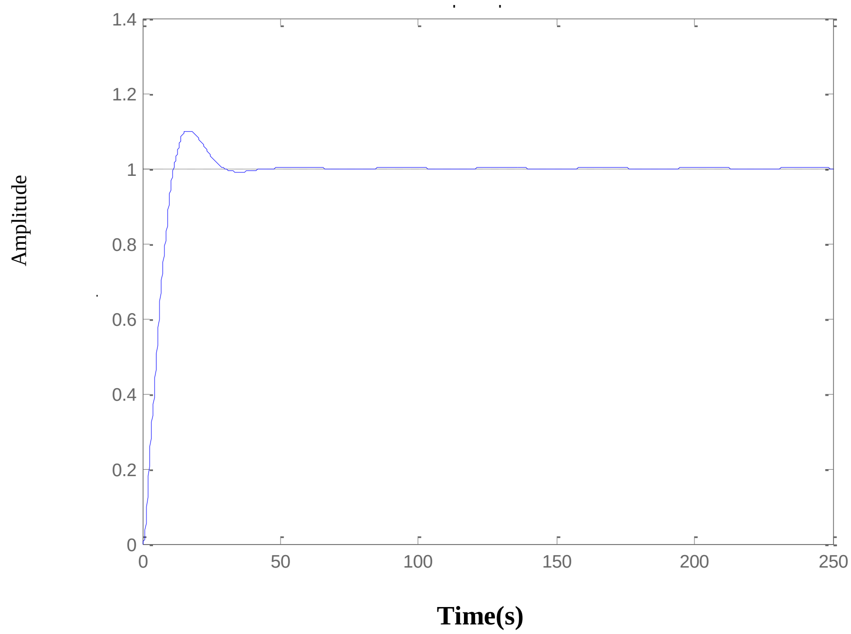

Figure 6. Step response of the close loop system with PID controller.

Table 1. Transient response specifications.

Controller	Percentage overshoot (%)	Rise time (s.)	Settling time (s.)	Delay time (s.)
PID	9.98	7.89	26.4	5.26
PI$^{\lambda}$D$^{\mu}$	4.4	4.72	151.71	3.21