University of Baghdad
College of Engineering
JOURNAL OF ENGINEERING

Journal of Engineering
journal homepage: www.joe.uobaghdad.edu.iq
Number 12 Volume 25 December 2019

Mechanical and Energy Engineering

Thermal Buckling of Angle-Ply Laminated Plates Using New Displacement Function

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ABSTRACT

Critical buckling temperature of angle-ply laminated plate is developed using a higher-order displacement field. This displacement field used by Mantari et al based on a constant ''m'', which is determined to give results closest to the three dimensions elasticity (3-D) theory. Equations of motion based on higher-order theory angle ply plates are derived through Hamilton' s principle, and solved using Navier-type solution to obtain critical buckling temperature for simply supported laminated plates. Changing (α_2/ α_1) ratios, number of layers, aspect ratios, E_1/E_2 ratios for thick and thin plates and their effect on thermal buckling of angle-ply laminates are studied in detail. It is concluded that, this displacement field produces numerical results close to 3-D elasticity theory with maximum discrepancy (7.4 %).

Keywords: thermal buckling, critical temperature, angle-ply plates, shear deformation theory

الانبعاج الحراري لصفائح طبقية غير متعامدة الزوايا باستخدام دالة ازاحة جديدة

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الخلاصة

درجة حرارة الانبعاج الحرج لصفائح طبقية مركبة غير متعامدة الزوايا تم تطويرها باستخدام مجال ازاحة ذات رتبة عالية. هذه الازاحة الجديدة اقترحت من J.L. Mantari et al الذي اعتمد على عامل ''m'' والتي تم ايجاد قيمتها لتعطي نتائج متطابقة مع الحل المرن ثلاثي الابعاد. تم اشتقاق مجموعة من المعادلات الاساسية لنظرية ثنائية الابعاد ذات رتبة عالية لصفائح طبقية غير متناظرة مكونة من طبقات غير متعامدة الزوايا من خلال مبدأ هاملتون (Hamilton's Principle). هذه المعادلات تم حلها باستخدام (Solution Navier's) لظروف حدودية بسيطة. تم دراسة شكل طور الانبعاج الحراري لصفائح طبقية مركبة غير متعامدة الزوايا مع نسبة التمدد الحراري (α_2/ α_1)، عدد الطبقات، نسبة الطول الى العرض، نسبة معامل المرونة لصفائح

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Peer review under the responsibility of University of Baghdad.
https://doi.org/10.31026/j.eng.2019.12.08

Article received: 12/6/2019
Article accepted: 22/9/2019
Article published: 1/12/2019

سميكة ورقيقة. وقد لوحظ ان هذه الازاحة تعطي نتائج متطابقة مع الحل المرن ثلاثي الابعاد وكان اكبر فرق (7.4 %) وكذلك النتائج الحاصلة من نظريات القص الاخرى وكان اكبر فرق (9.4 %).

الكلمات الرئيسية: الانبعاج الحراري، درجة الحرارة الحرجة، صفائح غير متعامدة، نظرية القص.

1. INTRODUCTION

Laminated composite plates are used widely in aeronautical, marine and mechanical industries as well as in other fields of modern engineering structures , those structure are often subjected to thermal load especially aircraft, launch vehicle and missiles structures, which may cause buckling of structure with certain boundary conditions, therefore there are many investigations about thermal buckling.

Thangaratnam, 1989. used finite element method using semiloof elements to analyze critical buckling temperature for composite laminates under thermal load. The equation of motion for critical temperature is obtained by equating the second variation of total potential energy to zero. Different boundary condition for cross-ply and angle-ply symmetric and antisymmetic stacking are analyzed. **Chang and Leu, 1991.** studied thermal buckling of antisymmetric angle-ply laminated simply supported subjected to uniform thermal load using higher order deformation theory which account for transverse shear and transverse normal strain to obtain exact-closed form solution. Obtained results are compared with first-order shear deformation theory and Reddy's higher-order shear deformation theory and showed surprising discrepancies exist. **Chen, et al, 1991,** implemented finite element method to analyze thermal buckling temperature of composite plates under uniform or nonuniform thermal load. Thermal-elastic Mindlin plate theory is used by which the transverse shear deformation and rotatory inertia were taken into account. **Meyers and Hyer, 1991,** used Rayleigh-Ritz formulation to obtain thermal buckling and post buckling response of symmetrically laminated composite plates. Two different laminates with two types of simply supported edges, fixed and sliding are investigated. Uniform temperature change along these laminates thickness is considered. **Noor and Burton, 1992,** Presented three-dimensional analytical solution for thermal buckling multilayered angle-ply composite plates with temperature-dependent thermo elastic properties. The temperature is assumed to be independent of the surface coordinates, but has symmetric variation along plate thickness. **Noor, et al , 1992,** studied buckling of laminated plates under combined thermal and axial loadings. Multi parameter-reduction method based on a first-order shear deformation theory, in connection with mixed finite-element is developed to study the effect of different lamination and material parameters on stability of the plate. **Noor, et al , 1992,** developed three-dimensional elasticity solutions for the critical buckling temperature of composite plates. The pre buckling deformations are taken into account. **Chen and Liu], 1993,** used first-order plate theory to analyze thermal buckling of angle-ply composite plates subjected to a uniform temperature change with Levy-type boundary conditions. **Prabhut and Dhanaraj, 1994,** analyzed thermal buckling of symmetric cross-ply and antisymmetric angle-ply laminated composite plates subjected to uniform temperature distribution using finite element method which based on the first order shear deformation theory. **Matsunaga, 2006,** investigated thermal buckling of angle-ply laminated composite and sandwich plates based on two-dimensional global higher order shear deformation theory. Fundamental governing equations are derived by the principle of virtual work and solved using power series expansion of continuous displacement components for simply supported laminated composite and sandwich plates. **Shiau et al , 2010,** studied thermal buckling behavior of composite laminated plates by using finite element method. The results indicate that the higher thermal buckling mode shapes are formed when the laminates produce higher bending rigidity along the fiber direction and higher in-plane compressive force in a direction perpendicular to the fiber direction. **Shi et al], 2010,** studied nonlinear thermal post buckling of antisymmetric angle-ply composite plates subjected on mechanical and thermal loads using finite element formulation. **Bourada et al , 2011,** used a new four-variable refined plate theory for thermal buckling analysis of functionally graded material (FGM) sandwich plates. The thermal loads are assumed as uniform, linear, and nonlinear temperature rises across the thickness direction. **Abdul-Majeed , 2011,** investigated thermal buckling of isotropic thermo elastic thin plates using governing differential equation and the Rayleigh-Ritz method. Three types of thermal distribution have been considered these are: uniform temperature, linear distribution and non-linear thermal distribution across thickness. **Naji , 2013,** investigated critical buckling temperature of cross-ply and angle-ply composite laminated plate using classical laminated and higher order shear deformation plate theory. Equations of motion are solved using Navier and Levy methods for symmetric and anti-symmetric laminated plates. **Singh, 2014,** presented thermal buckling behavior of laminated composite curved panel embedded with shape memory alloy fiber based on higher order shear deformation plate theory. Variational principle with finite element modeling under uniform temperature loading is used to obtain the responses. **Cetkovic and Gyorgy , 2016,** analyzed thermal buckling of angle-ply laminates using Generalized Layer wise Plate Theory. Element stiffness matrix and geometric stiffness matrix are derived based on finite element formulation. **Cetkovic , 2016,** studied thermal buckling of composite plates using new version of Layer wise. From the strong form, analytical solution is derived based on Navier's type, while the weak form is analyzed using the isoperimetric finite element approximation. **Chen, et al], 2016,** investigated Vibration and buckling behavior of initially stressed and thermally stressed composite plate using variation method. The temperature is assumed uniform and linearly distributed through the plate thickness. **Ounis and Belarbi , 2017,** studied the thermal buckling behavior of laminated plates with rectangular cutouts using classical plate theory as a base for finite element method. **Vescovini et al , 2017,** used Ritz-based variable-kinematic formulation to study thermal buckling of composite plates and sandwich panels. They represented structure by means of sub laminates. Critical temperatures obtained were for, with and without accounting for the pre-buckling. **Xing and Wang , 2017,** concerned the critical buckling temperature of functionally graded rectangular thin plates. Closed form solutions for the critical thermal parameter are obtained for the plate with different boundary conditions under uniform, linear and nonlinear temperature fields using separation-of-variable method.

In present work, critical temperature of simply supported composite plate is obtained using high order shear deformation theory of plate based on displacement field used by **Mantari et al , 2011** Effect of many thin and thick plate parameters, such as aspect ratio, E_1/E_2 ratio, α_2/α_1 ratio for antisymmetric angle ply are investigated.

2. DISPLACEMENT AND STRAIN

In present work, critical thermal temperature of simply supported angle ply laminated plate, based on new higher order theory is obtained using displacement field proposed by ,**Mantari et al , 2011:**

$$u(x,y,z) = u(x,y) - z\left(\frac{\partial w}{\partial x}\right) + f(z)\theta_1(x,y) \qquad (1)$$

$$v(x,y,z) = v(x,y) - z\left(\frac{\partial w}{\partial y}\right) + f(z)\theta_2(x,y) \qquad (2)$$

$$w(x,y,z) = w(x,y) \qquad (3)$$

Where:

$u_o(x,y), v_o(x,y), w_0(x,y), \theta_1(x,y), \theta_1(x,y), \theta_2(x,y)$ are the five unknown displacements of middle surface of the plate.

$f(z)$ is a shape functions to develop transverse shear strains and then stresses distribution along plate thickness.

For free boundary conditions at the top and bottom surfaces of the plate, the new proposed displacement field is**, Mantari et al, 2011:**

$$u(x,y,z) = u_o(x,y) + z\left(\frac{m\pi}{h}\theta_1 - \frac{\partial w}{\partial x}\right) + sin\frac{\pi z}{h}e^{m\cos\left(\frac{\pi z}{h}\right)}\theta_1 \quad (4)$$

$$v(x,y,z) = v_o(x,y) + z\left(\frac{m\pi}{h}\theta_2 - \frac{\partial w}{\partial y}\right) + sin\frac{\pi z}{h}e^{m\cos\left(\frac{\pi z}{h}\right)}\theta_2 \quad (5)$$

$$w(x,y,z) = w_0 \quad (6)$$

Where: $f(z) = sin\frac{\pi z}{h}e^{m\cos\left(\frac{\pi z}{h}\right)} + yz$, where, $y = \frac{\pi m}{h}$, $\mathrm{m} = \text{constant}$ (7)

The strain-displacement relations are, **Reddy, 2004.**

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},\ \varepsilon_{yy} = \frac{\partial v}{\partial y},\ \varepsilon_{zz} = \frac{\partial w}{\partial z} \quad (8,9 \text{ and } 10)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right) = \frac{1}{2}\gamma_{xy},\ \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial \mathrm{w}}{\partial \mathrm{x}}\right) = \frac{1}{2}\gamma_{xz} \quad (11 \text{ and } 12)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial \mathrm{w}}{\partial \mathrm{y}}\right) = \frac{1}{2}\gamma_{yz} \quad (13)$$

Substituting Eqs. (4-6) into Eqs. (8-13) to get the strain associated with the displacement field as follow:

$$\varepsilon_{xx} = \varepsilon^0_{xx} + z\varepsilon^1_{xx} + sin\frac{\pi z}{h}e^{m\,cos\left(\frac{\pi z}{h}\right)}\varepsilon^2_{xx} \quad (14)$$

$$\varepsilon_{yy} = \varepsilon^0_{yy} + z\varepsilon^1_{yy} + sin\frac{\pi z}{h}e^{m\,cos\left(\frac{\pi z}{h}\right)}\varepsilon^2_{yy} \quad (15)$$

$$\gamma_{xy} = \varepsilon^0_{xy} + z\varepsilon^1_{xy} + sin\frac{\pi z}{h}e^{m\,cos\left(\frac{\pi z}{h}\right)}\varepsilon^2_{xy} \quad (16)$$

$$\gamma_{xz} = \varepsilon^0_{xz} + (-m * sin^2(\frac{\pi z}{h}) + cos\frac{\pi z}{h})\frac{\pi}{h}e^{m\,cos\left(\frac{\pi z}{h}\right)}\varepsilon^3_{xz} \quad (17)$$

$$\gamma_{yz} = \varepsilon^0_{yz} + (-m * sin^2(\frac{\pi z}{h}) + cos\frac{\pi z}{h})\frac{\pi}{h}e^{m\,cos\left(\frac{\pi z}{h}\right)}\varepsilon^3_{yz} \quad (18)$$

Where:

$$\begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{m\pi}{h}\frac{\partial \Theta_1}{\partial x_1} - \frac{\partial^2 w}{\partial x^2} \\ \frac{m\pi}{h}\frac{\partial \Theta_2}{\partial y} - \frac{\partial^2 w}{\partial y^2} \\ \frac{m\pi}{h}\frac{\partial \Theta_2}{\partial x_1} + \frac{m\pi}{h}\frac{\partial \Theta_2}{\partial y} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^2 \\ \varepsilon_{yy}^2 \\ \gamma_{xy}^2 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \Theta_1}{\partial x} \\ \frac{\partial \Theta_2}{\partial y} \\ \frac{\partial \Theta_2}{\partial x} + \frac{\partial \Theta_1}{\partial y} \end{Bmatrix} \quad \text{(19,20 and 21)}$$

$$\begin{Bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{Bmatrix} = \begin{Bmatrix} m\frac{\pi}{h}\Theta_1 \\ m\frac{\pi}{h}\Theta_2 \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xz}^3 \\ \gamma_{yz}^3 \end{Bmatrix} = \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} \quad \text{(22 and 23)}$$

3. PRINCIPLES OF VIRTUAL WORK

The equations of motion will be derived depending on the new higher order theory using the Hamilton's principles ,**Reddy, 2004.**

$$0 = \int_0^t \delta U + \delta V \quad (24)$$

Where: δU is the virtual strain energy

$$\delta U = [\int_{\frac{-h}{2}}^{\frac{h}{2}} \{\int_{\Omega}^{k} \sigma_{xx}\delta\, \varepsilon_{xx}^k + \sigma_{yy}\delta\varepsilon_{yy}^k + \sigma_{xy}\delta\varepsilon_{xy}^k + \sigma_{yz}\delta\varepsilon_{yz}^k + \sigma_{xz}\delta\varepsilon_{xz}^k]\,\partial x \partial y\}\partial z] = 0 \quad (25)$$

$$\delta U = \int (N_1\delta\,\varepsilon_{xx}^0 + M_1\delta\varepsilon_{xx}^1 + P_1\delta\varepsilon_{xx}^2 + N_2\delta\varepsilon_{yy}^0 + M_2\delta\varepsilon_{yy}^1 + P_2\delta\varepsilon_{yy}^2 + N_6\delta\varepsilon_{Xy}^0 + M_6\delta\varepsilon_{Xy}^1$$

$$+P_6\delta\varepsilon_{Xy}^2 + Q_2\delta\varepsilon_{yz}^0 + k_2\delta\varepsilon_{yz}^3 + Q_1\delta\varepsilon_{xz}^0 + k_1\delta\varepsilon_{xz}^3 -)\partial x \partial y = 0 \quad (26)$$

Where, (N_i, M_i, P_i, Q_i and K_i) are the load results from the following integration:

$$(N_i, M_i, P_i) = \sum_{k=1}^{N}\int_{z^{k-1}}^{z^k} \sigma_i^k \left(1, z, sin\frac{\pi z}{h} e^{m\cos\left(\frac{\pi z}{h}\right)}\right) dz \qquad (i = 1,2,6)$$

$$(Q_1, K1) = \sum_{k=1}^{N}\int_{z^{k-1}}^{z^k} \sigma_5^k \left(1, \frac{\pi}{h}(-m * \, sin^2(\frac{\pi z}{h}) + cos\frac{\pi z}{h})e^{m\cos\left(\frac{\pi z}{h}\right)}\right) dz$$

$$(Q_2, K2) = \sum_{k=1}^{N}\int_{z^{k-1}}^{z^k} \sigma_4^k \left(1, \frac{\pi}{h}(-m * \, sin^2(\frac{\pi z}{h}) + cos\frac{\pi z}{h})e^{m\cos\left(\frac{\pi z}{h}\right)}\right) dz$$

Substituting equations of virtual strain (14-23) into Eq. (26) and in integrating by parts to relative virtual displacement (δu , δv , δw), then we get:

$$0 = -\int [\frac{\partial N_1}{\partial x}\delta u + \frac{m\pi}{h}\frac{\partial M_1}{\partial x}\delta\Theta_1 - \frac{\partial^2 M_1}{\partial x^2}\delta w + \frac{\partial P_1}{\partial x}\delta\Theta_1 + \frac{\partial N_2}{\partial y}\delta v + \frac{m\pi}{h}\frac{\partial M_2}{\partial y}\delta\Theta_2 - \frac{\partial^2 M_2}{\partial y^2}\delta w + \frac{\partial P_2}{\partial y}\delta\Theta_2 + \frac{\partial N_6}{\partial y}\delta u + \frac{\partial N_6}{\partial x}\delta v + \frac{m\pi}{h}\frac{\partial M_6}{\partial y}\delta\Theta_1 + \frac{m\pi}{h}\frac{\partial M_6}{\partial x}\delta\Theta_2 + 2\frac{\partial^2 M_6}{\partial x \partial y}\delta w + \frac{\partial P_6}{\partial y}\delta\Theta_1 + \frac{\partial P_6}{\partial x}\delta\Theta_2 - \frac{m\pi}{h}Q_1\delta\Theta_1 - \frac{m\pi}{h}Q_2 - K_1\delta\Theta_1 - K_2\delta\Theta_2]\partial x \partial y \quad (27)$$

The virtual work done by thermal applied load δV is:

$$\delta V=\int_{\Omega}\left\{N_x^T\,\delta\left(\frac{\partial\ w}{\partial x}\right)^2+N_y^T\,\delta\left(\frac{\partial\ w}{\partial y}\right)^2+N_{xy}^T\,\delta\left(\frac{\partial\ w}{\partial x}\right)\times\left(\frac{\partial\ w}{\partial y}\right)\right\}dxdy \tag{28}$$

4. EQUATIONS OF MOTION

The Euler-Lagrange equations are determined by substituting Eqs. (27 – 28) into Eq. (24) to derive equations of motion as follows:

$$\delta u:\ \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0 \tag{29}$$

$$\delta v:\ \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0 \tag{30}$$

$$\delta w:\ \frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+N_x^T\left(\frac{\partial^2 w}{\partial x^2}\right)+N_y^T\left(\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{31}$$

$$\delta\Theta_1:\ \frac{m\pi}{h}\frac{\partial M_x}{\partial x}+\frac{m\pi}{h}\frac{\partial M_{xy}}{\partial y}+\frac{\partial P_x}{\partial x}+\frac{\partial P_{xy}}{\partial y}-\frac{m\pi}{h}Q_x-K_x=0 \tag{32}$$

$$\delta\Theta_2:\ \frac{m\pi}{h}\frac{\partial M_y}{\partial y}+\frac{m\pi}{h}\frac{\partial M_{xy}}{\partial x}+\frac{\partial P_y}{\partial y}+\frac{\partial P_{xy}}{\partial x}-\frac{m\pi}{h}Q_y-K_y=0 \tag{33}$$

The plane stress reduced stiffness Q_{ij} is:

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\ ,\ Q_{11}=\frac{E_2}{1-\nu_{12}\nu_{21}},Q_{66}=G_{12}\ ,\ Q_{44}=G_{23}\ ,\ Q_{55}=G_{13} \tag{34}$$

Where:

G_{12}, G_{23} and G_{13}= shear modulus of plate in planes 12, 23 and 13 respectively.

E1 and E_2 = Young's modulus in directions 1 and 2 of the plate.

υ_{12} and υ_{21} are poison's ratio in directions 12 and 21 respectively.

The transformed stress-strain relation of an orthotropic lamina in a plane state of stress is:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11}Q_{12}Q_{16}\\ Q_{12}Q_{22}Q_{26}\\ Q_{16}Q_{26}Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}-\alpha_{xx}\Delta T\\ \varepsilon_{yy}-\alpha_{xx}\Delta T\\ \gamma_{xy}-2\alpha_{xy}\Delta T\end{Bmatrix},\begin{Bmatrix}\sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}=\begin{bmatrix}Q_{44} & Q_{45}\\ Q_{45} & Q_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{35}$$

The force results are:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11}A_{12}A_{16} & B_{11}B_{12}B_{16} & E_{11}E_{12}E_{16} \\ A_{12}A_{22}A_{26} & B_{12}B_{22}B_{26} & E_{12}E_{22}E_{26} \\ A_{16}A_{26}A_{66} & B_{16}B_{26}B_{66} & E_{16}E_{26}E_{66} \\ B_{11}B_{12}B_{16} & D_{11}D_{12}D_{16} & F_{11}F_{12}F_{16} \\ B_{12}B_{22}B_{26} & D_{12}D_{22}D_{26} & F_{12}F_{22}F_{26} \\ B_{16}B_{26}B_{66} & D_{16}D_{26}D_{66} & F_{16}F_{26}F_{66} \\ E_{11}E_{12}E_{16} & F_{11}F_{12}F_{16} & H_{11}H_{12}H_{16} \\ E_{12}E_{22}E_{26} & F_{12}F_{22}F_{26} & H_{12}H_{22}H_{26} \\ E_{16}E_{26}E_{66} & F_{16}F_{26}F_{66} & H_{16}H_{26}H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \\ \varepsilon_x^1 \\ \varepsilon_y^1 \\ \varepsilon_{xy}^1 \\ \varepsilon_x^2 \\ \varepsilon_y^2 \\ \varepsilon_{xy}^2 \end{Bmatrix} \tag{36}$$

$$\begin{Bmatrix} Q_x \\ Q_y \\ K_x \\ K_y \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} & J_{44} & J_{45} \\ A_{45} & A_{55} & J_{45} & J_{55} \\ J_{44} & J_{45} & L_{44} & L_{45} \\ J_{45} & J_{55} & L_{45} & L_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \\ \gamma_{yz}^3 \\ \gamma_{xz}^3 \end{Bmatrix} \tag{37}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \sum_{k=1}^{N} \int_{z^k}^{z^{k+1}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \Delta T dz \tag{38}$$

Where: $A_{ij} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\, dz \quad i = (1,2,4,5,6)$ (39)

$$(B_{ij}, D_{ij}, E_{ij}) = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}(z, z^2, sin(\frac{\pi z}{h}) e^{m\, cos\, (\frac{\pi z}{h})}, \quad (i, j=1, 2, 6) \tag{40}$$

$$(F_{ij}, H_{ij}) = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}\, (sin\left(\frac{\pi z}{h}\right) e^{m\, cos\, (\frac{\pi z}{h})} * z, sin^2(\frac{\pi z}{h})\, e^{2m\, cos\, (\frac{\pi z}{h})}) \qquad (i, j=1, 2, 6) \tag{41}$$

$$J_{ij} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij} \frac{\pi}{h} e^{m\, cos\, (\frac{\pi z}{h})} (-m * sin^2(\frac{\pi z}{h}) + cos\frac{\pi z}{h}) dz \tag{42}$$

$$L_{ij} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij} \left(\frac{\pi}{h}\right)^2 e^{2m\, cos\, (\frac{\pi z}{h})} \left(-m * sin^2(\frac{\pi z}{h}) + cos\frac{\pi z}{h}\right)^2 dz \quad i,j = (4,5) \tag{43}$$

5. NAVIER'S SOLUTION

The generalized displacements are expanded in a double trigonometric series in terms of unknown parameters in Navier's method. To satisfy the boundary conditions of the problem, the restricted choice of the function in the series is selected.

Assuming the following displacements form to satisfied simply supported boundary conditions:

Reddy , 2004.

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}\, sin(\alpha x)\, cos(\beta y) \tag{44}$$

$$v(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}\ cos(\alpha x)\, sin(\beta y) \tag{45}$$

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\, sin(\alpha x)\, sin(\beta y) \tag{46}$$

$$\theta_1(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ \theta_{1_{mn}}\, sin(\alpha x)\, cos(\beta y) \tag{47}$$

$$\theta_2(x,y,t) = \sum_{m=1}^{\infty}\ \sum_{n=1}^{\infty}\ \theta_{2_{mn}}\, cos(\alpha x)\, sin(\beta y) \tag{48}$$

Where: $\alpha = \frac{m\pi}{h}$, $\beta = \frac{n\pi}{h}$ $(U_{mn}, V_{mn}, W_{mn},\ \theta_{1_{mn}}, \theta_{2_{mn}})$, are arbitrary constants.

6. EIGNVALUE PROBLEM

Equations of motion Eqs. (29-33) can be expressed in terms of displacements by substituting the force and moment resultants from Eqs. (36 - 38) and using Eqs. (44-48), result an eignvalue as following:

$$\left(\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ & c_{22} & c_{23} & c_{24} & c_{25} \\ & & c_{33-\alpha^2 N_x^T - \beta^2 N_y^T - \alpha\beta N_{xy}^T} & c_{34} & c_{35} \\ & & & c_{44} & c_{45} \\ & & & & c_{55} \end{bmatrix}\right)\{d\} = 0 \tag{49}$$

Where:$\{d_{ij}\} = \{U_{mn}, V_{mn}, W_{mn},\ \theta_{1_{mn}}, \theta_{2_{mn}}\}$

And C_{ij} is the element of stiffness, from which the critical buckling temperature for the plate can be obtained.

7. RESULTS AND DISCUSSION

Using above analytical solutions of the HOSDT based on displacement field given by [Mantari et al], 2011, a computer program is built using MATLAB15 programming for thermal buckling of laminated angle ply composite plates. The parametric effect of side to-thickness ration (a/h), plate, aspect ratio a/b, modulus ratio E_1/E_2 and thermal expansion coefficient ratio ($\alpha_2/\ \alpha_1$) on critical buckling temperature of laminated composite plates are analyzed. To verify the suggested above solution, obtained results are compared with three dimension elasticity theory proposed by ,**Noor, 1992**, which give good agreement with maximum discrepancy (7.4 %) for ten layers of anti symmetric angle ply with different thickness ratio (a/h) and different angles as shown in **Table 1.** Also, as compared with first-order thick plate theory proposed by ,**Chen and Liu], 1993,** for anti symmetric six layers angle ply [45/-45] plate, as listed in **Table 2.** with maximum discrepancy (9.4 %).

Changing of aspect ratio effect on critical buckling temperature of ten layers laminated thick and thin antisymmetric plates, are listed in **Table 3.** Which show that critical temperature decreases as aspect ratio (a/b) increases, also it decrease with increasing (a/h) ratio which effected critical temperature larger than (a/b) ratio. Different critical thermal buckling modes for plates with different aspect ratio for $[45\ -45]_2$ angle-ply square plate are shown in **Figs. 1-3.** For these three figures the normalized critical temperature is ($T^*cr = T^*\ \alpha_1 * 10 * (b/h)^2$) and material properties are, E_1/E_2=25, $G_{12}=G_{13}=0.5\ E_2$, $G_{23}=0.2\ E_2$, ($\alpha_2/\ \alpha_1$=3), $\nu_{12}=\nu_{13}=\nu_{23}=0.25$.

Tables 4. and 5. show another comparison with , **Chen and Liu , 1993,** for antisymmetric laminated square thick and moderately thick plate (a/h =10 and a/h = 20) for different aspect ratio (b/a), angle orientation (30, 45 and 60) and number of layer (2, 4 and 8) which give closed results with maximum discrepancy (3.5 %).

Table 6. show the effect of changing (E_1/E_2) on critical temperature for four and eight layers anti symmetric angle ply plates for different thickness ratio (a/h), since stiffness increase when increasing orthotropic ratio therefore normalized critical temperature increase.

Effect of thermal expansion coefficient ratio (α_2/α_1) on critical buckling temperature of four layer laminated thick and thin plates [with different (a/h) ratio], are listed in **Table 7.** as expected critical temperature decrease when (α_2/α_1) increase and (a/h) increase since stiffness decrease when plate become thinner.

Table 1. Normalized critical temperature ($T_{cr} = T^* \alpha_0$) for angle-ply square plate, E_1/E_2=15, $G_{12}=G_{13}$=0.5 E_2, G_{23}=0.3356 E_2, ν_{12}=0.3, α_2/α_1=0.015, N = 10.

a/h	References	Tcr			
		θ = 0° (m,n) (1,2)	**θ** = 15° (m,n) (1,2)	**θ** = 30° (m,n) (1,1)	**θ** = 45° (m,n) (1,1)
4	Present	0.1872	0.2221	-	-
	Noor, 1992	0.1777	0.2087	-	-
	Discrepancy %	5.3	6.4	-	-
5	Present	0.1504	0.1849	0.2554 (1,2)	-
	Noor, 1992	0.1436	0.1753	0.2377 (1,2)	-
	Discrepancy %	4.7	5.4	7.4	-
10	Present	0.05917	0.08124	0.1125	0.12259
	Noor, 1992	0.05782	0.07904	0.1100	0.1194
	Discrepancy %	2.3	2.7	2.2	2.67
20	Present	0.01752	0.02552	0.03472	0.03844

	Noor, 1992	0.01739	0.02528	0.03446	0.03810
	Discrepancy %	0.74	0.94	0.75	0.89
100	Present	0.0007465	0.001115	0.001502	0.001674
	Noor	0.0007463	0.001115	0.001502	0.001674
	Discrepancy %	0.026	0	0	0

Table 2. Normalized critical temperature (Tcr = T*1000* α_0) for angle-ply square plate, E_1=21, E_2= E_3=1.7, G_{12}=G_{13}=0.65, G_{23}= 0.639, ν_{12}= ν_{13} =0.21, α_2 =16, α_1=-0.21, $[45\ -45]_3$.

a/h	Chen and Liu, 1993	Present work	Discrepancy %
5	21.3622	19.5369	9.3
8	12.7542	11.6360	9.6
10	9.2963	8.4905	9.4
15	4.7885	4.3836	9.2
20	2.8523	2.6142	9.1
30	1.3234	1.2142	8.9
40	0.7560	0.6939	8.9
50	0.4874	0.4474	8.9
80	0.1919	0.1762	8.9
100	0.1230	0.1129	8.9

Table 3. Normalized critical temperature ($Tcr = T^* \alpha_0$) for anti-symmetric layers angle-ply square plate, $E_1/E_2=15$, $G_{12}=G_{13}=0.5 E_2$, $G_{23}=0.3356 E_2$, $\nu_{12}=0.3$, $\alpha_2/ \alpha_1=0.015$, [45 -45], N=10.

a/b	b/h			
	5	10	20	100
1	0.2734	0.1225	0.03844	0.00167
2	0.1961	0.0738	0.02121	0.00089
3	0.1670	0.0588	0.01643	0.00068
4	0.1547	0.0527	0.01454	0.00060

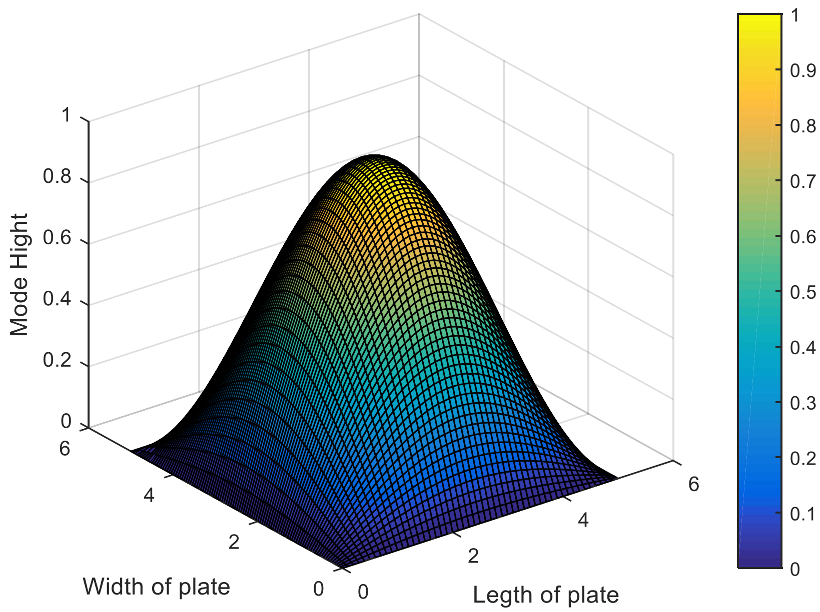

Figure 1. Normalized Thermal Buckling mode for antisymmetric angle-ply square plate, mode(m=1,n=1), No. of layers=4, a/h=5, a/b=1.

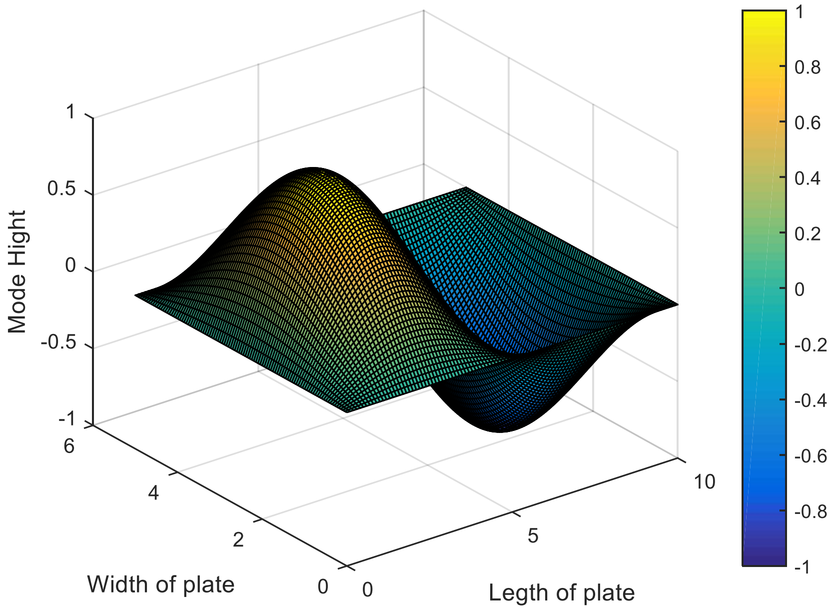

Figure 2. Normalized Thermal Buckling mode for antisymmetric angle-ply square plate, mode(m=2,n=1), No. of layers=4, b/h=5, a/b=2.

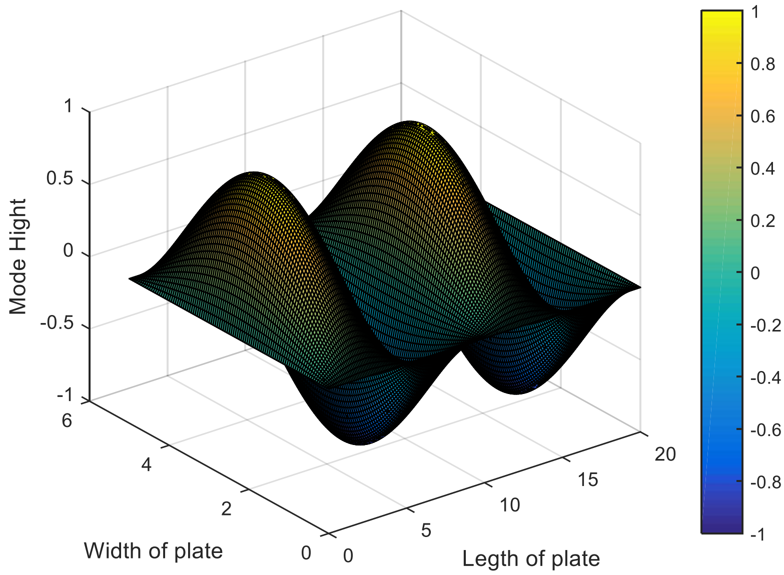

Figure 3. Normalized Thermal Buckling mode for antisymmetric angle-ply square plate, mode(m=4,n=1), No. of layers=4, b/h=5, a/b=4.

Table 4. Normalized critical temperature [$T_{cr} = T^* \alpha_1 * 10 * (b/h)^2$] for angle-ply plate with $a/h=10$, $E_1/E_2=25$, $G_{12}=G_{13}=0.5\ E_2$, $G_{23}=0.2\ E_2$, $\nu_{12}=0.25$, $\alpha_2/\alpha_1=3$.

b/a	Angle	No. of layers	Tcr		Discrepancy %
			Chen and Liu, 1993	Present	
1	30	2	4.0600	4.1897	3.1
		4	7.1345	6.9253	3.02
		8	7.7267	7.5925	1.7
	45	2	4.2070	4.3568	3.5
		4	7.6605	7.4158	3.2
		8	8.3010	8.1519	1.8
	60	2	4.0600	4.1897	3.1
		4	7.1345	6.9253	3.02
		8	7.7267	7.5925	1.7
2	30	2	2.6431	2.7035	2.2
		4	4.9757	4.8559	2.4
		8	5.4530	5.370	1.5
	45	2	2.4942	2.5479	2.1
		4	4.8401	4.7158	2.6
		8	5.3240	5.2326	1.7
	60	2	2.3437	2.3771	1.4
		4	4.3038	4.1977	2.5
		8	4.7266	4.6373	1.9

Table 5. Normalized critical temperature [Tcr = T* α_1*10*$(b/h)^2$] for angle-ply square plate for a/h=20 , E_1/E_2=25, $G_{12}=G_{13}$=0.5 E_2, G_{23}=0.2 E_2, ν_{12}=0.25, α_2/α_1=3.

Angle	No. of layers	Tcr		Discrepancy %
		Chen and Liu, 1993	Present	
30	2	4.7891	4.8461	1.1
	4	9.7366	9.5515	1.9
	8	10.8736	10.6972	1.6
45	2	4.9920	5.0503	1.1
	4	10.7342	10.5094	2.1
	8	12.0354	11.8277	1.7
60	2	4.7891	4.8461	1.1
	4	9.7366	9.5515	1.9
	8	10.8736	10.6972	1.6

Table 6. Normalized critical temperature (Tcr = T* α_0) for different (E_1/E_2) of angle-ply square plate, $[45\ -45]_2$, mode (m=1, n=1), $G_{12}=G_{13}$= 0.5 E_2, G_{23}=0.3356 E_2, ν_{12}= 0.3, α_2/α_1=0.015, .

a/h	N=4				N=8			
	E_1/E_2				E_1/E_2			
	10	15	25	40	10	15	25	40
5	0.2244	0.2524	0.2748	0.2766	0.2402	0.2711	0.2953	0.2972
10	0.08793	0.11009	0.1389	0.1597	0.09635	0.1215	0.1535	0.1759
20	0.02567	0.03395	0.04709	0.06065	0.0284	0.0380	0.0532	0.06872
100	0.00108	0.001468	0.002128	0.002915	0.001208	0.00165	0.00243	0.0033

Table 7. Effect of (α_2/α_1) on normalized critical temperature Tcr for $[45/-45]_2$ angle-ply square plate, mode (m=1, n=1), $E_1/E_2=25$, $G_{12}=G_{13}=0.5\ E_2$, $G_{23}=0.2\ E_2$, $\nu_{12}=\nu_{13}=\nu_{23}=0.25$.

a/h		Tcr
	(α_2/α_1)	N = 4
4	2	0.016466
	4	0.014978
	6	0.013736
	8	0.012685
	10	0.011783
10	2	0.007784
	4	0.007080
	6	0.006493
	8	0.005996
	10	0.005570
100	2	0.000127
	4	0.000115
	6	0.000106
	8	0.000098
	10	0.000091

8. CONCLUSIONS

Thermal buckling analysis of angle ply composite thick and thin plates is developed by using new displacement field developed by ,**Mantari et al, 2011,** but with changing a parameter ''m'', to 'm=.05' then obtained results are well agree with 3D elasticity theory solution and other plate solution methods. As expected critical temperature is decreased as thickness ratio and aspect ratio increased while the buckling temperature decreases with the increase of thermal expansion coefficient ratio $\alpha_2/\ \alpha_1$ and is larger for thick, than thin laminates.
thermal buckling mode of simply supported angle-ply plate does not change according to this displacement field.

REFERENCES

- Abdul-Majeed, Jweeg and Jameel, 2011, *Thermal buckling of rectangular plates with different temperature distribution using strain energy method*, Journal of Engineering, 5 (17):1047-1065.

- Bourada, Tounsi, Houari and Bedia, 2011, *A new four-variable refined plate theory for thermal buckling analysis of functionally graded sandwich plates*, Journal of Sandwich Structures and Materials, 14(1), 5-33.

- Chang and Leu, 1991, *Thermal buckling analysis of antisymmetric angle-ply laminates based on a higher-order displacement field*, Composites Science and Technology, 41, 109-128.

- Chen, Lin and Chen, 1991, *Thermal buckling behavior of thick composite laminated plates under nonuniform temperature distribution*, Computers & Structures Vol. 41, No. 4, pp. 637-645.

- Chen and Liu, 1993, *Thermal buckling of antisymmetric angle-ply laminated plates- an analytical Levy-type solution*, Journal of Thermal Stresses, 16:4, 401-419.

- Chen, Chen and Lin, 2016, *Thermally induced stability and vibration of initially stressed laminated composite plates*, Mechanika, Volume 22(1),51-58.

- Cetkovic and Gyorgy, 2016, *Thermo-Elastic stability of angle-ply laminates application of layerwise finite elements*, Structural Integrity And Life, Vol. 16 No 1 pp. 43–48.

- Cetkovic, 2016, *Thermal buckling of laminated composite plates using layerwise displacement model*, Composite Structures, 142 May, 238-253.

- Mantari, Oktem and Soares, 2011, *Static and dynamic analysis of laminated composite and sandwich plates and shells by using a new higher-order shear deformation theory*, Composite Structures, 2011 94 37–49.

Matsunaga, 2006, *Thermal buckling of angle-ply laminated composite and sandwich plates according to a global higher-order deformation theory*, Composite Structures, 72, 177–192.

- Meyers and Hyer, 1991, *Thermal Buckling And Postbuckling of Symmetrically laminated Composite Plates*, Journal of Thermal Stresses, 14:4, 519-540.

- Naji and Nsaif, 2013, *Buckling Analysis of Composite Plates Under Thermo-Mechanical Loading*, Journal of Al Rafidain University College, Issue No. 32.

- Noor and Burton, 1992, *Three-Dimensional Solutions for the Thermal Buckling and Sensitivity Derivatives of Temperature-Sensitive Multilayered Angle-Ply Plates*, Journal of Applied Mechanics (Transactions of the ASME), 59:848-856.

- Noor, and Burton, 1992, *Thermomechanical Buckling of Multilayered Composite Plates*, Journal of Engineering Mechanics, 118, 351-366.

- Noor, and Peters, 1992, *Three-Dimensional Solutions For Thermal Buckling of Multilayered Anisotropic Plates*, Journal of Engineering Mechanics, Vol. 118, No.4, April.

- Ounis and Belarbi, 2017, *On the thermal buckling behavior of laminated composite plates with cut-outs*, Journal Of Applied Engineering Science & Technology, 3(2) 63-69.

- Prabhut and Dhanaraj, 1994, *Thermal Buckling of Laminated Composite Plates*, Computers & Structures, Vol. 53, No. 5. pp. 1193-1204.

- Prof. Dr. Jameel, Dr. Sadiq and M.Sc. student Nsaif 2012, *Buckling analysis of composite plates under thermal and mechanical loading*, Journal of Engineering, Volume 18 ,No. 12, pp1365-1390.

- Reddy, 2004, *Mechanics of laminated composite plates and shells*, second edition CRC press.

- Shi, Lee and Mei, 1999, *Thermal Postbuckling of Composite Plates Using The Finite Element Modal Coordinate Method*, Journal of Thermal Stresses, 22,595-614.

- Shiau, Kuo and Chen, 2010, *Thermal buckling behavior of composite laminated plates*, Composite Structures, 92, 508–514.

- Singh and Panda 2014, *Thermal Buckling Analysis of Laminated Composite Shell Panel Embedded with Shape Memory Alloy Fibre under TD and TID*, Thesis Submitted to National Institute of Technology, Rourkela, June.

- Thangaratnam, Palaninathan and Ramachandran, 1989, *Thermal Buckling of composites laminated plates*, Computers & structures, Vol. 32, No. 5, PP. 1117-1124.

- Vescovini, D'Ottavio and Polit, 2017, *Thermal Buckling Response of Laminated and Sandwich Plates using Refined 2-D Models*, Composite Structures, vol. 176, 15 September, Pages 313-328.

- Xing and Wang, 2017, *Closed Form Solutions for Thermal Buckling of Functionally Graded Rectangular Thin Plates*, Applied Sciences, 7, 1256.