University of Baghdad
College of Engineering
JE
JOURNAL OF ENGINEERING

Journal of Engineering
journal homepage: www.joe.uobaghdad.edu.iq
Number 8 Volume 25 August 2019

Mechanical and Energy Engineering

Buckling Analysis of Laminated Composite Plate with Different Boundary Conditions using modified Fourier series

Dr. Widad I. Majeed*
Assistant professor
College of Engineering-University of Baghdad
Baghdad, Iraq
wedad.ibrahim@coeng.uobaghdad.edu.iq

Zainab Abdul Kareem Abed
M.Sc. student
College of Engineering-University of Baghdad
Baghdad, Iraq
zainabkareem2158@gmail.com

ABSTRACT

Buckling analysis of a laminated composite thin plate with different boundary conditions subjected to in-plane uniform load are studied depending on classical laminated plate theory; analytically using (Rayleigh-Ritz method). Equation of motion of the plates was derived using the principle of virtual work and solved using modified Fourier displacement function that satisfies general edge conditions. The eigenvalue problem generated by using Ritz method, the set of linear algebraic equations can be solved using MATLAB for symmetric and anti-symmetric, cross and angle-ply laminated plate considering some design parameters such as aspect ratios, number of layers, lamination type and orthotropic ratio. The results obtained gives good agreement with those published by other researchers.

Keywords: buckling load, Rayleigh-Ritz method, different edge conditions, and composite laminated plate.

تحليل الانبعاج لصفائح مركبة طبقية مع ظروف أسناد عامة بأستخدام متسلسلة فورير المتطورة

أ.م.د وداد ابراهيم مجيد
أستاذ مساعد
جامعة بغداد-كلية الهندسة
قسم الهندسة الميكانيكية

زينب عبد الكريم عبد
طالب ماجستير
جامعة بغداد-كلية الهندسة
قسم الهندسة الميكانيكية

الخلاصة

تمت دراسة الانبعاج للصفائح الرقيقة المركبة المكونة من طبقات المعرضة لأحمال منتظمة ضمن مستوي الصفيحة باستخدام نظرية الصفائح الكلاسيكية(CLPT). في الجانب النظري التحليلي تم اشتقاق معادلات الحركة باستخدام طريقة رتز للحصول على مجموعة معادلات متجانسة وحلها باستخدام متسلسلة فورير المتطورة وضعت هذه الدراسة المنهج العام لاشتقاق مجموعة كاملة من الدوال التي يمكن تطبيقها على حالة الحدودية المختلفة وحلها كقيم eigen بأستخدام برنامج (Matlab) لالواح متعامدة متماثلة وغير متماثلة ولالواح بطبقات ذات زوايا متماثلة وغير متماثلة.
مع الأخذ بنظر الاعتبار تغيير في بعض معايير التصميم مثل شروط الحدود نسبة الارتفاع وزاوية التصفيح ونسبة السماكة ونسبة ال orthotropy حيث تم مقارنة النتائج مع باحثين اخرين واعطت تقارب جيد جدا.
الكلمات الرئيسية: حمل الانبعاج ، نظرية القص ذات الرتبة العالية ، الالواح الطبقية المركبة.

*Corresponding author
Peer review under the responsibility of University of Baghdad.
https://doi.org/10.31026/j.eng.2019.08.1

Article received: 7/10/2018
Article accepted: 18/11/2018

1.INTRODUCTION

Any material involving two or more constituents by different properties and distinction boundary among the constituents can be known as a composite material. The main components of composite material are matrix and fibers, **Singh, 2012.** Many advantages for composites modified, such as specific strength can range as high as four times of those of high strength steel compounds, and for specific modulus, the rate can be as high as seven times those to titanium, aluminum, steel compounds. The fiber reinforced composite is used in the format of a comparatively thin plate so consequently the load carrying ability of composite laminated plate with buckling taken into account considered by researchers with different loading and edge conditions. The main essential problems of laminated plates even now are the buckling problems, and that has attracted the attention of many researchers in the present. Different solution methods are used to solve the buckling analysis by researchers such as the Navier method, the Levy method, the Ritz technique, and the finite element method. The Levy solutions can be advanced for plates with two opposite edges simply supported and the other two boundaries having clamped or free edge conditions. The Navier solutions can be improved for a composite plate when all four boundaries are simply supported. Ritz technique and finite element method are used to determine approximate solutions for different edge conditions, **Reddy, 2003**.

Many researchers have studied the critical buckling load of composite plates**. Zhong, and Gu, 2007** investigated the buckling load factor of simply supported symmetric cross-ply (0-90) rectangular and square plates subjected to unidirectional linearly varying in-plane loads (uniform and non-uniform). The exact solution is developed to obtain the buckling behavior based on the first order shear deformation theory (FSDT). The results were verified by comparing the present work with the computer code ABAQUS. This work investigated the effect of aspect (a/b), thickness-to-width (h/b) and the modulus(E_1/E_2) ratio on the buckling load factor. **Shufrin, et al., 2008** presented the buckling analysis of symmetric cross and angle ply under uniaxial and biaxial compression load laminated rectangular composite plates with various edge conditions. The multi-term extended Kantorovich method was used to reduce the partial differential buckling equations to ordinary differential equations (ODE). The equation of motion is derived using the principle of virtual work. The resulting eigenvalue problem is solved using the exact element method for buckling analysis. The accuracy of the method is examined over the numerical analysis of different rectangular plates with various in-plane loads (uniaxial or biaxial) and edge conditions. **Kuo, and Shiau, 2009** showed the buckling analysis and vibration of composite plates with different fibre spacing. The finit element method (FEM) is used. The results showed the buckling load and natural frequencies increased efficiently when the fibers distributed in the middle section of the plate. The fibers distributed in the external section of the plate increase the buckling load. **Kim, et al., 2009** presented two variable refined plate theories for buckling analysis of composite and isotropic plate. The equation of motion is derived by using the principle of virtual work. The Navier technique is used to investigate the solution of simply supported composite plate applying to in-plane load. Numerical results determined by the theory was present and are compared with classical laminated plate theory (CLPT) and first order shear deformation theory (FSDT) solutions. They concluded the used theory does not need shear correction factor, simple to use and also similar to (FSDT). **Kumar Panda, and Ramachandra, 2010** presented the buckling analysis of laminated plates for different edge conditions and applying to non-uniform in-plane loading. The theory used is higher order shear deformation theory (HSDT). For the edge conditions, suitable beam functions were used as displacement field in Galerkin's method. The buckling load is determined from calculating the related linear eigenvalue problem. The obtained results are compared with the results obtained by other researchers and with the numerical results using ANSYS to verify them. **Thai, and Kim, 2011** investigated the buckling analysis of orthotropic Levy plate based on two variable refined plate theory. Comparison studies are performed to verify the validity of the present results. In addition, the closed-form solutions of orthotropic plate based on CPT are also generated for the verification purpose. The effects of boundary condition, loading condition, and variation of modulus ratio, aspect ratio, and thickness ratio on the critical buckling loads of orthotropic plates are studied and discussed in details. **Mohan Kumar, et al., 2013** showed the effect of the length-to-thickness *(a/t)* ratio, the aspect ratio *(a/b)*, the fiber orientation on the buckling load for the glass epoxy laminated plate in clampthe ed-free-clamped-free configuration by finite element method (FEA). Buckling analysis was carried out on the laminated plate both; numerically and experimentally; for the two various geometrical configurations to predict the buckling load and the obtained results were compared with the finite element method, and give good agreement. **Sayyad, and Ghugal, 2014** studied the natural frequencies and critical buckling loads of composite plate by using (ESDT). The deformation model contains exponential terms in addition to (CLPT) terms. The governing equations are derived based on the Hamilton's principle. The Navier type solution is used for solving this equation of simply supported square plates. The Navier solution for laminated plate based on (HSDT), (TSDT), (FSDT) and (CLPT) for verification purpose. The results are determined by using analytic and then a comparison was made with the existing higher order theories to analyzing the vibration and buckling behavior of composite plates. **Widad,** and **Firas, 2015** studied buckling and free vibration analysis of composite thin plates subjected to various distributed loads using (CLPT). Also, they investigated the effect the buckling loads for composite plates with various combinations parameter such as edge conditions on the natural frequencies and also determined this buckling loads. The transverse deflection is considered with specific suitable functions depending on the type of the chosen boundary conditions applied to the edges which may be simply supported, clamped or free, the edge conditions proposed here are all edges simply, all edges clamped, two edges simply and other clamped, two edges simply and other free and two edges clamped and other free; the chosen functions are sin-cosine combinations. Analytical investigation is presented using the Ritz method for homogeneous equations eigenvalue problems. This study accounts the effect of the boundary conditions, aspect ratio, load ratio, and lamination angle. The results are verified by comparing them to results determined by (FEM) using ANSYS, form experimental results and that determined by other researchers. **Osman, et al., 2017** presented the Buckling analysis of symmetric cross – ply rectangular laminates under uniaxial and biaxial compression. They used finite element analysis based on classical laminate theory. The effect of boundary condition, aspect ratio$(\frac{a}{b})$ and elastic modulus ratio on buckling load is explained. It is found that as the plate becomes more restrained its resistance to buckling increases. Also, the critical buckling load decreases when the modulus ratio increases and becomes almost constant for higher values of the elastic modular ratio.

In current work, Rayleigh-Ritz technique is used to investigate the critical buckling of uniaxial and biaxial compression loads for angle and cross-laminated composite plate under different edge conditions, using unified modified Fourier function.

2. THEORETICAL ANALYSIS:

2.1 Buckling Analysis of Laminated Plates:

The governing equation is derived by using CPLT, **Sayyad, 2014:**

$$D_{11}\frac{\partial^4 w}{\partial x^4}+(2D_{12}+4D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2} \quad (1)$$

Where stress resultants are expressed in displacement form from below:

$$\left(M_x, M_y, M_{xy}\right)=\int_{-h/2}^{h/2}\left(\sigma_x, \sigma_y, \sigma_{xy}\right)z\,dz=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\left(\sigma_x, \sigma_y, \sigma_{xy}\right)z\,dz \quad (2)$$

Integrating Eq. (2) through the thickness of the plate, the stress resultant is associated with the displacement (w) by the relatives:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}=\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (3)$$

$$D_{ij}=\left[\bar{Q}_{ij}\right]\int_{\frac{h}{2}}^{\frac{-h}{2}} z^2\,dz \quad (4)$$

Where $\bar{Q}_{ij}$ transformed reduced stiffness and D_{ij} bending stiffness matrix **, Reddy, 2004.**
The twisting moments and bending, transversal shear forces can be written in terms of the displacement function as, **Henry Khov, 2009.**

$$M_x=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2} \quad (5)$$

$$M_y=-D_{22}\frac{\partial^2 w}{\partial y^2}-D_{12}\frac{\partial^2 w}{\partial x^2} \quad (6)$$

$$M_{xy}=-2D_{66}\frac{\partial^2 w}{\partial x\partial y} \quad (7)$$

$$Q_x=-D_{11}\frac{\partial^3 w}{\partial x^3}-(D_{12}+4D_{66})\frac{\partial^3 w}{\partial y^2\partial x} \quad (8)$$

$$Q_y=-D_{22}\frac{\partial^3 w}{\partial y^3}-(D_{12}+4D_{66})\frac{\partial^3 w}{\partial x^2\partial y} \quad (9)$$

For a flexible restricted rectangular plate shown in **Fig.1**, the boundary conditions are:

$$k_{x0}w=Q_x \quad K_{x0}\frac{\partial w}{\partial x}=-M_x \text{ .. at x=0} \quad (10\text{-}11)$$

$$k_{x1}w=-Q_x \quad K_{x1}\frac{\partial w}{\partial x}=-M_x \text{ .. at x=a} \quad (12\text{-}13)$$

$$k_{y0}w = Q_y \quad K_{y0}\frac{\partial w}{\partial y} = -M_y \text{ .. at y=0} \tag{14-15}$$

$$k_{y1}w = -Q_y \quad K_{y1}\frac{\partial w}{\partial y} = -M_y \text{ .. at y=b} \tag{16-17}$$

Where k_{y0}, k_{y1} *and* k_{x0}, k_{x1} are the transitional stiffness of spring, K_{y0}, K_{y1} *and* K_{x0}, K_{x1} are the rotations stiffness of spring at y=0 and b (x=0 and a), respectively. Eq. (10)-(17) express a set of different edge conditions from which, the entire classic homogeneously boundary conditions can be direct gotten by as putting the constants of spring equalize to an very small or large number.

From Eq. (5-17), the edge conditions can be finally written as follows:

$$k_{x0}w = -D_{11}\frac{\partial^3 w}{\partial x^3} - (D_{12} + 4D_{66})\frac{\partial^3 w}{\partial y^2 \partial x} \tag{18}$$

$$k_{x1}w = D_{11}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 4D_{66})\frac{\partial^3 w}{\partial y^2 \partial x} \tag{19}$$

$$K_{x0}\frac{\partial w}{\partial x} = D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} \tag{20}$$

$$K_{x1}\frac{\partial w}{\partial x} = D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} \tag{21}$$

And similarly, the other four equations in the y-direction are found.

As mentioned for many plates and shell researches that exact solution for plate or shell with general boundary conditions is not available, so the Ritz method is used to get an approximate solution from Hamilton's equation:

$$\delta \iiint (U - W) = 0 \tag{22}$$

Where U is the strain energy, W is potential energy due to the external forces. δ the random variation.

Where

$$U = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^b\int_0^a \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right) dx\, dy\, dz + \text{Elastic energy of springs at edges}$$

$$W = \frac{1}{2}\int_0^b\int_0^a \left[N_x\left(\frac{\partial w}{\partial x}\right)^2\right] dx\, dy + \frac{1}{2}\int_0^b\int_0^a \left[N_y\left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy \tag{23}$$

Substituting Eqs. (2- 3),(18-21) in Eq.(23), we get:

$$\mathrm{U}-\mathrm{W}=\frac{1}{2}\int_0^b\int_0^a\left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2+\ 4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]dx\,dy+\frac{1}{2}\int_0^b\left[k_{x0}w^2+K_{x0}\left(\frac{\partial w}{\partial x}\right)^2\right]_{x=0}dy+\frac{1}{2}\int_0^b\left[k_{x1}w^2+K_{x1}\left(\frac{\partial w}{\partial x}\right)^2\right]_{x=a}dy+\frac{1}{2}\int_0^a\left[k_{y0}w^2+K_{y0}\left(\frac{\partial w}{\partial y}\right)^2\right]_{y=0}dx+\frac{1}{2}\int_0^a\left[k_{y1}w^2+K_{y1}\left(\frac{\partial w}{\partial y}\right)^2\right]_{y=b}dx-\frac{1}{2}\int_0^b\int_0^a\left[N_x\left(\frac{\partial w}{\partial x}\right)^2\right]dx\,dy-\frac{1}{2}\int_0^b\int_0^a\left[N_y\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy \quad (24)$$

2.2 Admissible functions:

In the Rayleigh-Ritz technique, the allowable functions play an essential part. The products of the beam functions are regularly selected as the allowable functions, and the displacement function can be accordingly expressed as, **W.L. Li, 2004.**

$$w(x,y)=\sum_{m,n=1}A_{mn}X(x)\,Y(y) \quad (25)$$

Where X(x), Y(y) are the specific variables for beams that include similar edge conditions in the (y, x) direction, correspondingly.

Though functions of the beam can be in general achieved as a linear combination of hyperbolic and trigonometric functions, they involve some unknown parameters that must be determined from the boundary conditions. Accordingly, then, every boundary condition essentially leads to a various set of beam functions. In actual applications, this is disadvantageous, beside the tediousness of determining the essential functions for a different boundary beam. To avert this difficulty, developed Fourier series method has been suggested for beams with an arbitrary boundary at both ends in which the characteristic functions are written in the form of, **Li,2000.**

$$w(x)=\sum_{m=0}^{\infty}a_m\cos\lambda_{am}x+P(x)\qquad\left(\lambda_{am}=\frac{m\pi}{a}\right),\qquad 0\leq x\leq a. \quad (26)$$

P (x) is the function in Eq. (26) considers an arbitrarily continued function that, in any case of edge conditions, is constantly selected to satisfy the equations as follows:

$$P'''(0)=W'''(0)=\alpha_{0,}\qquad P'''(a)=W'''(a)=\alpha_{1,} \quad (27\text{-}28)$$

$$P'(0)=W'(0)=\beta_{0,}\quad and\qquad P'(a)=W'(a)=\beta_{1,} \quad (29\text{-}30)$$

P (x) is here inserted to take care of the latent discontinuities of the function of displacement and its derivative at endpoints. Accurately, previously it is known that the smoothest a periodical function, the quicker its Fourier extension convergence. Thus, adding of the P (x) will have two instantaneous interests: (1) the series of Fourier extension is presently agree with any edge conditions, and (2) the solution of the series of Fourier and its accurateness of convergences.

P (x) has just been realized as a continuous function that satisfies Eq. (27) - (30), the function P (x) format is not a worry regarding the convergences of the series solution. Therefore, it can be chosen in any required formula. Like a substantiation, supposes the P (x) is a function of polynomial,

$$P(x) = \sum_{n=0}^{4} C_n \, P_n\left(\frac{x}{a}\right), \tag{31}$$

Where $P_n(x)$ is the Legendre function of order n , C_n is the coefficient of extension.

It is clarified that P (x) desires to be minimum a 4th polynomial to satisfy Eq.(27) - (30) jointly. Substituting Eq. (31) into Eq. (27) - (30) results in

$$C_3 P_3'''(0) + C_4 P_4'''(0) = a^3 \alpha_0, \tag{32}$$

$$C_3 P_3'''(1) + C_4 P_4'''(1) = a^3 \alpha_1, \tag{33}$$

$$C_1 P_1'(0) + C_2 P_2'(0) + C_3 P_3'(0) + C_4 P_4'(0) = a\beta_0, \tag{34}$$

$$C_1 P_1'(1) + C_2 P_2'(1) + C_3 P_3'(1) + C_4 P_4'(1) = a\beta_1, \tag{35}$$

From the last equations, the constants C_n $(n = 1, 2, 3, 4)$, are straight acquired in terms of the boundary constants, α_0, α_1, β_0, and β_1. since the constant C_0 doesn't really seem in Eq. (32)-(35), it can be a random number theoretically. For example, C_0 is here chosen to satisfy, **Li, 2002.**

To find p(x) from **Li, 2002** by integration the P_1''' and P_2' then will obtained eq (39)

$$\int_0^a P(x)\, dx = 0 \tag{36}$$

The last appearance for the P(x) can be shown as

$$\text{P(x)} = \zeta_a(x)^T \bar{\boldsymbol{\alpha}} \tag{37}$$

Where

$$\bar{\boldsymbol{\alpha}} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^T \tag{38}$$

and

$$\zeta_a(x)^T = \begin{Bmatrix} -(15x^4 - 60ax^3 + 60a^2x^2 - 8a^4)/360a \\ (15x^4 - 30a^2x^2 + 7a^4)/360a \\ (6ax - 2a^2 - 3x^2)/6a \\ (3x^2 - a^2)/6a \end{Bmatrix} \tag{39}$$

The results in Eq. (37) - (39) are already derived from an additional simple but a little common approach, **Li, 2004.**

To obtain the unknown of edge constants, $\alpha_o, \alpha_1, \beta_o$ andβ_1, the substitution of Eq.(26) , (37) into the edge conditions Eq.(18)-(23) that results in

$$\bar{\boldsymbol{\alpha}} = \sum_{m=0}^{\infty} H_a^{-1} Q_{am} a_m \tag{40}$$

Where

$$H_a = \begin{bmatrix} 1+\frac{8k_{x0}a^3}{360D_{11}} & \frac{7k_{x0}a^3}{360D_{11}} & \frac{-k_{x0}a}{3D_{11}} & \frac{-k_{x0}a}{6} \\ \frac{7k_{x1}a^3}{360D_{11}} & 1+\frac{8k_{x1}a^3}{360D_{11}} & \frac{-k_{x1}a}{3D_{11}} & \frac{-k_{x1}a}{6} \\ \frac{a}{3} & \frac{a}{6} & \frac{K_{x0}}{D_{11}}+\frac{1}{a} & \frac{-1}{a} \\ \frac{a}{6} & \frac{a}{3} & \frac{-1}{a} & \frac{K_{x1}}{D_{11}}+\frac{1}{a} \end{bmatrix} \quad (41)$$

and

$$Q_{am} = \left\{(-1)\frac{k_{x0}}{D_{11}} \quad (-1)^m\frac{k_{x1}}{D_{11}} \quad -\lambda_{am}^2 \quad (-1)^m\lambda_{am}^2\right\}^T \quad (42)$$

It must be reminded that the matrix Ha will become single for a total free Beam. Though, this problem can be got over to some extent by artificially connecting one or more springs with the smallest stiffness to the ends of a beam. It has been shown in**, W.L. Li, 2002.** Though the matrix might be ill-conditioned in such a treating and the natural frequencies can still be accurately calculated for a completely free beam. However, the characteristic functions are very suitable for this special case and can be easily used as the allowable functions in the Rayleigh-Ritz technique.

By using Eqs. (37) and (40), Eq. (29) becomes as:

$$w(x) = \sum_{m=0}^{\infty} a_m\varphi_m^a(x) \quad (43)$$

Where

$$\varphi_m^a(x) = \cos\lambda_{am}x + \zeta_a(x)H_a^{-1}Q_{am} \quad (44)$$

Eq. (25) can be consequently rewritten as:

$$w(x,y) = \sum_{m,n=0}^{\infty} A_m\varphi_m^a(x)\varphi_n^b(y) \quad (45)$$

Where:

$$\varphi_n^b(y) = \cos\lambda_{bn}y + \zeta_b(y)H_b^{-1}Q_{bn} \quad (46)$$

The terms for $\zeta_b(y), H_b and\ Q_{bn}$ can be, correspondingly, obtained from Eqs. (39), (41) and (43) by easily changing the x- regarding parameters by the y- regarding.

2.3 Determination of Critical Buckling Loads:

Consider an orthotropic laminated plate, the material directions of width coincide with the plate directions. The plate is subjected to biaxial in-plane compressive forces $N_x and N_y$ along the both sides of edges (x) and (y), respectively.

The total mechanical energy can be written in the following expressions:

$$E = \frac{1}{2}\int_0^b\int_0^a \left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + D_{66}\left(\frac{2\partial^2 w}{\partial x \partial y}\right)^2 + 2\left(D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{16}\frac{\partial^2 w}{\partial x^2}\frac{2\partial^2 w}{\partial x \partial y} + D_{26}\frac{\partial^2 w}{\partial y^2}\frac{2\partial^2 w}{\partial x \partial y}\right)\right] dx\, dy + \frac{1}{2}\int_0^b \left[k_{x0}w^2 + K_{x0}\left(\frac{\partial w}{\partial x}\right)^2\right]_{x=0} dy + \frac{1}{2}\int_0^b \left[k_{x1}w^2 + K_{x1}\left(\frac{\partial w}{\partial y}\right)^2\right]_{x=a} dy + \frac{1}{2}\int_0^a \left[k_{y0}w^2 + K_{y0}\left(\frac{\partial w}{\partial y}\right)^2\right]_{y=0} dx \frac{1}{2}\int_0^a \left[k_{y1}w^2 + K_{y1}\left(\frac{\partial w}{\partial y}\right)^2\right]_{y=b} dx - \frac{1}{2}\iint \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy$$
(47)

By minimizing the total mechanical energy with respect to A:

$$\frac{\partial E}{\partial A_{mn}} = 0 \quad (48)$$

Eq. (48) will result in a set of algebraic equations like the following:

$f(A_{mn}, N_{cr}) = 0 \quad for\ buckling\ problem$ (49)

Solving Eq. (48) as an Eigenvalue problem to obtain the following:

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,(m*n)} \\ \vdots & \ddots & \vdots \\ a_{(m*n),1} & \cdots & a_{(m*n),(m*n)} \end{bmatrix} \begin{Bmatrix} A_{11} \\ \vdots \\ A_{mn} \end{Bmatrix} = 0 \quad (50)$$

Eq. (50) equating it to zero to get critical buckling load N_{cr}. For different edge conditions and M &N more than one, the solution becomes very complicate and needs a program to findN_{cr}. For numerical study, ANSYS (15.0) programming is used.

3-RESULTS AND CONCLUSIONS

In present work modified Fourier series is used for the first time to obtain critical buckling load of laminated plates, for verification all results are compared with others obtained by many researchers, as shown below:

3-1-Results

Eigenvalue problem obtained by Ritz method is solved by using MATLAB (version15) to investigate the buckling load of the composite laminated plate (CLP) with elastic edge condition. A four-letter symbol describes the plates for example SCSC denotes a plate with simply supported edge at y=b, y=0, clamped at x=a, x=0. Numerical results of the orthotropic plate are compared with those found by, **Widad,** and **Firas, 2015** and **Shufrin, et al., 2008,** which shows good agreement between the results. As shown in **Tables 1 and 2**, the laminated plates considered here have symmetric cross-ply [0 90 0]and angle$[30\ -30\ \ 30]$-ply and subjected to uniaxial compression. It is noticed that the clamped edge along two or four edges can hold buckling load more than a plate with simply supported boundary conditions, especially for the case where plate SFSF in **Table 1,** because of the high stiffness due to the boundary. When the plate is simply supported or mixed with free edges, it is weak to hold large loads compared with clamped plates, because of the its lower stiffness. The results for the composite plate with various edge conditions showed, stacking sequence, (a\b) and modulus ratio giving a good agreement when compared with **Reddy, 2003** and **Shufrin, et al., 2008. Tables 3, 4, and 5,** show that the buckling behavior of laminated plate is the same as obtained by other researchers. When changing some design parameters such as modulus ratio and aspect ratio $(\frac{a}{b})$ simply supported anti-symmetric cross-ply and angle-ply buckling load in present work is presented in **Table 6,** also results give good agreement with those obtained by **Reddy, 2003,** while **Tables 7 and 8** give critical biaxial buckling load for simply supported (SSSS) cross-ply and angle-ply (symmetric and anti-symmetric) and for different aspect ratio with changing modulus ratio, again these results agree in value or behavior with those obtained **Reddy, 2003,** while **Table 9** gives biaxial critical buckling load for (0 90 0) laminated plate under different boundary conditions, also when compared with those obtained by **Reddy, 2003,** give good agreement in value and behavior.

3.2. CONCLUSIONS

This study investigated the buckling analysis of a composite laminated plate. Several assumptions are made to solve the buckling problem.

The results are obtained basically by using analytic analysis and then compared with the results determined by other researchers; the comparison shows good agreement between them.

The results yielded the following conclusions:

1- Modified Fourier series is an efficient function for critical buckling analysis of laminated plates with general edge conditions.
2- The aspect ratio is inversely proportional with the buckling as proved by other methods used in other researchers when changing some design parameters such as modulus ratio and aspect ratio$(\frac{a}{b})$.
3- The edge conditions affect the buckling load. Clamped boundary conditions show high stiffness, results in high buckling load. Clamped edges made the plate carries load larger than the simply supported edges; where the buckling load for SFSF laminated plate under uniform load, is less by 80% than the critical load of CCCC.
4- The uniaxial compression can carry buckling load higher than the biaxial compression .
5- The angle-ply can carry buckling load higher than the cross-ply for high modulus ratios.

NOMENCLATURE

Symbols	Description	Units
a, b	plate length and width, respectively	M
E_1,E_2, E_3	modulus of elasticity in 1, 2, and 3 directions ,respectively	Gpa

E_x	modulus of elasticity in x directions (45^o directions)	Gpa
E, E_c	total mechanical and kinetic energies of a system	N.m
G_{12}, G_{23}, G_{13}	shear modulus in plane 1-2, 2-3, 1-3 , respectively	Gpa
h	thickness of the laminate	Mm
h_t, h_b	thickness at the top and bottom of the laminate	Mm
h_k, h_{k-1}	distances from the reference plane of the laminate to the two surfaces of the kth ply	Mm
I_o	second mass moment of inertia	Kg.m^2
i, j	components of series	-----
κ_x, κ_y, κ_{xy}	curvatures of the reference plane of the plate	-----
k	layer number	-----
L	total number of layers in the laminate	-----
M_x, M_y, M_{xy}	bending and twist moments per unit length acting on a laminate	N.m/m
Q_x , Q_y	shear force	N
K_{x0} , K_{x1}	rotational stiffness at x = 0 and a, respectively	N.m/rad
K_{y0} , K_{y1}	rotational stiffness at y = 0 and b, respectively	N.m/rad
k_{x0} , k_{x1}	translational stiffness at x = 0 and a, respectively	N/m
k_{y0} , k_{y1}	translational stiffness at y = 0 and b, respectively	N/m
M, N	upper limits of double series	-----
t	time	sec.
a_m	expansion or Rayleigh-Ritz coefficient	-----
U	strain energy of deformation	N.m
V_c	elastic potential energy	N.m
u, v, w	displacements in x, y, z directions, respectively	M
u_o, v_o, w_o	displacements of the reference surface in the x, y, z	M

	directions, respectively	
$W(X)$	flexural displacement of a beam	M
$w(x, y)$	flexural displacement of a plate	M
$w(x, y, t)$	dynamic displacement	M
V	volume of object	m^3
V_f, V_m, V_v	volume fractions of fiber, matrix, and voids respectively	-----
v_c, v_f	volumes of the composite, fiber, respectively	m^3
W	total work done	N.m
P(x)	a simple polynomial function	-----
$X_m(x)$, $Y_n(y)$	beam characteristic function	-----

ABBREVIATION LIST

Abbreviation	Description
FEM	finite element method
CLPT	classical laminated plate theory
HSDPT	higher order shear deformation plate theory
FSDT	first-order shear deformation Theory
ESL	equivalent single layer
FRPC	fiber reinforced polymeric composites
ODE	ordinary differential equations
UAC	uniaxial compression
BAC	biaxial compression
PS	pure shear
UAS	uniaxial compression and shear
BAS	biaxial compression and shear
ESDT	exponential shear deformation theory

REFERENCES:

- Singh, Yogesh. 2012. *Free Vibration Analysis of the Laminated Composite Beam with Various Boundary Conditions, A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF Master of Technology In Machine Design and Analysis*.
- (J._N._Reddy Mechanics of Laminated Composite Plates and Shells Theory and Analysis Second Edition Crc_Press_2003.Pdf n.d.)
- Zhong, Hongzhi, and Chao Gu. 2007. Buckling of Symmetrical Cross-Ply Composite Rectangular Plates under a Linearly Varying in-Plane Load. *Composite Structures,* Vol. 80, No. 1, PP. 42–48.
- I. Shufrin, O. Rabinovitch, and M. Eisenberger. 2008, Buckling of symmetrically laminated rectangular plates with general boundary conditions - A semi analytical approach, *Compos. Struct.*, Vol. 82.4, PP. 521-531.
- Kuo, Shih Yao, and Le Chung Shiau. 2009. “Buckling and Vibration of Composite Laminated Plates with Variable Fiber Spacing.” *Composite Structures*, Vol.90, No. 2, pp.196–200.
- Kim Seung-Eock., Huu-Tai Thai., and Jaehong Lee., 2009, Buckling Analysis of Plates using the Two Variable Refined Plate Theory, Thin-Walled Structures Vol. 47.4, PP. 455-462.
- Kumar Panda, Sarat, and L. S. Ramachandra. 2010. Buckling of Rectangular Plates with Various Boundary Conditions Loaded by Non-Uniform Inplane Loads. *International Journal of Mechanical Sciences,* Vol.52, No. 6, pp. 819–28.
- Thai, Huu Tai, and Seung Eock Kim. 2011. Levy-Type Solution for Buckling Analysis of Orthotropic Plates Based on Two Variable Refined Plate Theory. *Composite Structures* Vol. 93, No. 7, PP. 1738–46.
- Mohan Kumar, M, Colins V Jacob, and M Nagabhushana. 2013. Volume-02, Issue-07. *American Journal of Engineering Research (AJER)*: PP. 33–40.
- Sayyad. Atteshamuddins, and Y. M. Ghugal, 2014 *Buckling and free vibration analysis of orthotropic plates by using exponential shear deformation theory*, Latin American Journal of Solids and Structures vol. 11.8, pp. 1298-1314.
- Assist. Prof. Dr. Widad I. Majeed and Firas H. Taya., 2015, *Stability and Dynamic Analysis of Laminated Composite Plates, Journal of Engineerin*g, Vol.21, No.8, PP. 139-159.
- Osman, Mahmoud Yassin., and Osama Mohammed Elmardi Suleiman., 2017, *Buckling analysis of thin laminated composite plates using finite element method,* IJERAT 3.
- Henry Khov., Wen, L. Li., and Ronald F. Gibson., 2009, *An Accurate Solution Method for the Static and Dynamic Deflections of Orthotropic Plates with General Boundary Conditions,* Composite Structures, Vol. 90, PP. 474–481.
- J.N. REDDY., 2004, *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis,* 2nd ed.; CRC Press LLC.

- W.L. Li., 2002, *Reply to: discussion on, Free Vibrations with Beam with General Boundary Condition, Journal of Sound and Vibration*, Vol. 257, PP. 593–595.
- W.L. Li ., 2004, *Vibration Analysis of Rectangular Plates with General Elastic Boundary Supports, Journal of Sound and Vibration*, Vol. 273 , PP. 619–635.
- Prof. Dr. Adnan Naji Jameel., Dr. Ibtehal Abbas Sadiq., and M.SC. Hasanian Ibraheem Nsaif., 2012, *Buckling Analysis of Composite Plates under Thermal and Mechanical loading, Journal of Engineering*, Vol. 18, No. 12, PP. 1365-139.

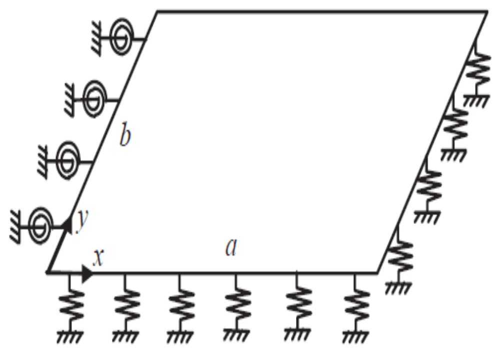
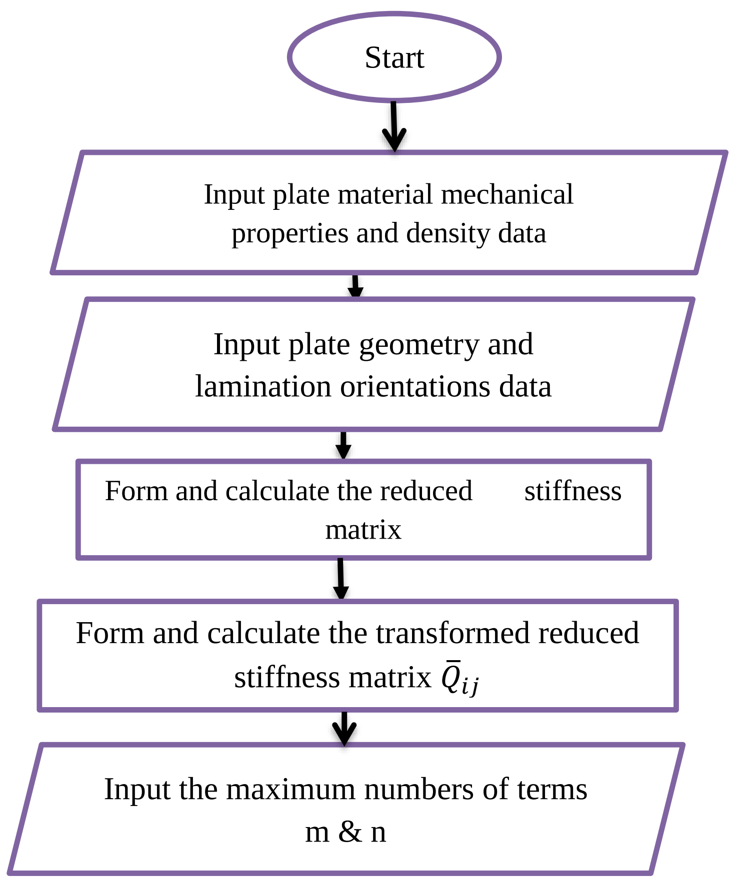


Figure 1. Elastic restrained edges for a rectangular plate.



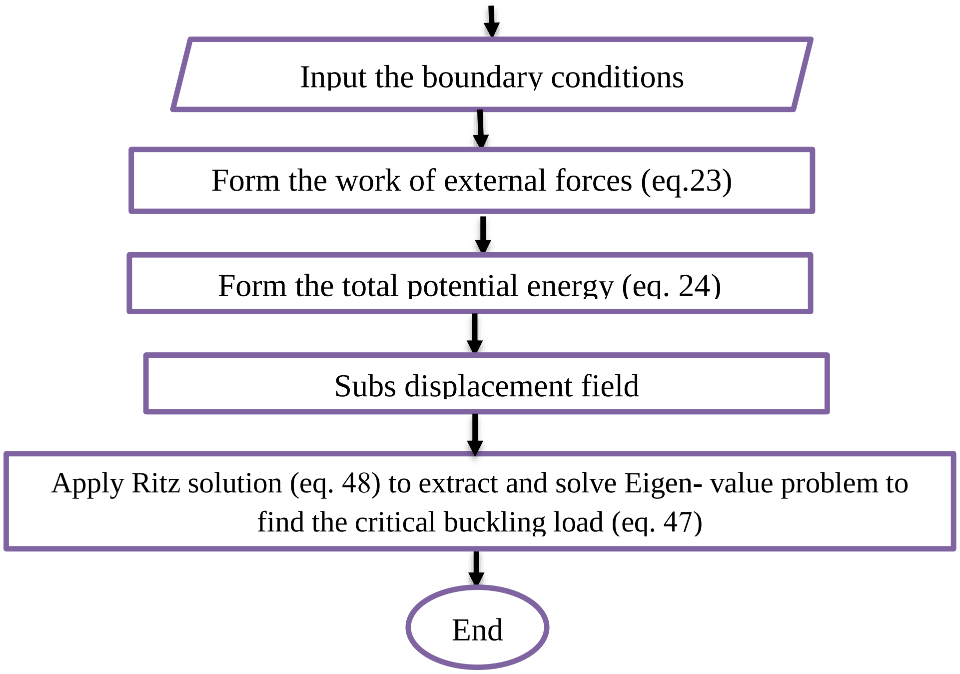


Figure 2. Block diagram of determining critical buckling using MATLAB R2015a.

Table 1. Non-dimensional buckling load($\bar{N} = N_{cr}b^2/E_2h^3$), for [0 90 0] plates of different Boundary conditions, ($E_1/E_2 = 10, G_{12} = 0.6E_2, \nu_{12} = 0.25, a = b$).

References	Type of boundary conditions				
	SSSS	CCCC	SCSC	FSFS	FCFC
Present work	11.550	40.38	35.900	8.049	32.544
Firas, 2015	11.491	40.507	36.255	7.991	32.982

Table 2. Non-dimensional buckling load ($\bar{N} = N_{cr}\,12(1 - \nu_{12}\nu_{21})/E_1\,h^3$), for [30 -30 30] plates of different boundary conditions, ($E_1/E_2 = 2.45, G_{12} = 0.48E_2, \nu_{12} = 0.23$ a=b).

References	Type of boundary conditions			
	SSSS	CCCC	CSCS	SCSC
Present work	25.77	66.81	47.33	39.32
Shufrin, 2008	26.67	65.26	49.18	40.93
Discrepancy%	3.4	2.3	3.9	4

Table 3. Non-dimensional buckling load($\overline{N} = N_{cr}b^2/D_{22}\pi^2$), for [0 90 90 0] plates (SSSS) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

References	a/b	E_1/E_2=5	10	20	25	40
Present work	0.5	13.94	18.225	22	23.1	25
Reddy, 2004		13.9	18.126	21.87	22.87	24.59
Present work	1	5.66	6.353	7	7.13	7.5
Reddy, 2004		5.65	6.347	6.96	7.12	7.4
Present work	1.5	5.238	5.28	5.317	5.326	5.34
Reddy, 2004		5.233	5.27	5.31	5.318	5.33

Table 4. Non-dimensional buckling load($\overline{N} = N_{cr}b^2/D_{22}\pi^2$), for $[0\ 90]_{2s}$ laminated plates (CCCF) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

E_1/E_2	References	a/b =1	1.5	2
3	Present work	6.7	3.48	2.47
	Shufrin, 2008	6.4	3.3	2.34
	Discrepancy%	4.7	5	5
10	Present work	8.08	3.96	2.6
	Shufrin, 2008	7.84	3.78	2.48
	Discrepancy%	2.9	4	4

Table 5. Non-dimensional buckling load($\overline{N} = N_{cr}b^2/D_{22}\pi^2$), for $[0\ 90]_{2s}$ laminated plates (CSCS) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

$\frac{E_1}{E_2}$	References	a/b =1	1.5	2
3	Present work	6.671	6.379	6.12
	Shufrin, 2008	6.659	6.295	5.84
	Discrepancy%	0.179	1.3	4.5
10	Present work	6.584	6.096	5.71
	Shufrin, 2008	6.557	6.056	5.46
	Discrepancy%	0.41	0.656	4.3

Table 6. Non-dimensional buckling load($\bar{N} = N_{cr}b^2/E_2h^3$), for anti-symmetric laminated plates (SSSS) with the effect of different modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

Ply Orientations	References	E_1/E_2=10	25	40
$[0\ 90]_4$	Present work	11.174	23.523	35.874
	Reddy	10.864	22.622	34.381
	Discrepancy%	2.7	3.8	4.1
$[45\ -45]_4$	Present work	18.2	42.81	67.38
	Reddy	17.637	41.16	64.68
	Discrepancy%	3	3.8	4

Table 7 . Non-dimensional buckling load under Biaxial compression($\bar{N} = N_{cr}b^2/D_{22}\pi^2$), for [0 90 90 0] plates (SSSS) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

References	a/b	E_1/E_2=5	10	20	25	40
Present work	0.5	11.132	12.718	13.946	14.271	14.831
Reddy,2004		11.12	12.694	13.922	14.248	14.766
Present work	1	2.827	3.176	3.484	3.566	3.706
Reddy,2004		2.825	3.174	3.481	3.562	3.702
Present work	1.5	1.611	1.625	1.635	1.638	1.642
Reddy,2004		1.610	1.624	1.634	1.636	1.641

Table 8. Non-dimensional buckling load under Biaxial compression($\bar{N} = N_{cr}b^2/E_2h^3$), for anti-symmetric laminated plates (SSSS) with the effect of different modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

Ply Orientations	References	E_1/E_2=10	25	40
$[0\ 90]_4$	Present work	5.587	11.761	17.937
	Reddy, 2004	5.432	11.311	17.190
	Discrepancy%	2.7	3.8	4.1
$[45\ -45]_4$	Present work	9.108	21.447	33.788
	Reddy, 2004	8.813	20.578	32.343
	Discrepancy%	3	3.8	4

Table 9. Non-dimensional critical buckling load under Biaxial compression($\bar{N} = N_{cr}b^2/E_2h^3$), of symmetric cross ply (0/90/0) square plates for various boundary conditions and modulus ratios, ($G_{12} = 0.6E_2, \nu_{12} = 0.25$).

E_1/E_2	References	**SSSS**	**SCSS**	**SCSC**	**SFSF**
10	Present- work	5.751	9.363	13.487	1.105
	Reddy, 2004	5.746	9.353	13.468	1.123
20	Present- work	9.631	13.984	22.038	1.407
	Reddy, 2004	9.591	14.026	21.709	1.420